Equation de Boltzmann:

$$\partial_t f(v_1,t) = g_2(\sigma)\,\sigma^{d-1}\int_{\mathbb{R}^d} dv_2 \int d\hat\sigma\; \Theta(\hat\sigma\cdot v_{12})(\hat\sigma\cdot v_{12})\left[\frac{1}{\alpha^2} f(v_1^{**},t)\, f(v_2^{**},t) - f(v_1,t)\, f(v_2,t)\right] + \frac{\xi_0^2}{2}\frac{\partial^2}{\partial v_1^2} f(v_1,t)\,, \qquad (1)$$

où $g_2(\sigma)$ est la fonction de corrélation de deux particules au contact. Pour $d=3$, on a $g_2(\sigma)=\frac{2-\eta}{2(1-\eta)^3}$, $\eta=\pi n\sigma^3/6$. Ici ξ_0 est un coefficient de diffusion dans l'espace des vitesses. Sa valeur est fixée par le fait que dans l'état stationnaire, la température doit être constante. On en tire au premier ordre en a_2 [1]

$$\xi_0^2 = (1-\alpha^2)\,\frac{S_d\, g_2\, n\sigma^{d-1}}{d\sqrt{\pi}}\left(\frac{T_0}{m}\right)^{3/2}\left(1+\frac{3}{16}a_2(0)\right), \qquad (2)$$

ou de façon équivalente, la température cinétique stationnaire est:

$$T_0 = m\left[\frac{d\,\xi_0^2\sqrt{\pi}}{(1-\alpha^2)\,S_d\, g_2\, n\sigma^{d-1}\,(1+3/16\;a_2(0))}\right]^{2/3}. \qquad (3)$$

Solution de scaling:

$$f(\underline{v},t) = \frac{n}{v_T^d(t)}\,\tilde f(\underline{c},t) \;;\quad c = v/v_T(t) \qquad (4)$$

$$\tilde f(\underline{c},t) = \mathcal{M}(c)\left[1+\sum_{i=2}^{\infty} a_i(t)\, S_i(c^2)\right] \;;\quad \mathcal{M}(c) = \pi^{-d/2}e^{-c^2} \qquad (5)$$

$$S_n(x) = L_n^{d/2-1}(x) = \sum_{j=0}^{n}(-1)^j\binom{n+d/2-1}{n-j}\frac{x^j}{j!} \qquad (6)$$

; $L_n^\alpha(x)$: polynôme de Laguerre généralisé.

On introduit la forme d'échelle dans l'équation de Boltzmann (1):

$$\partial_t f = n\tilde f\,\partial_t v_T^{-d} + n v_T^{-d}\,\partial_t \tilde f = -d\,\frac{n}{v_T^{d+1}}\frac{dv_T}{dt}\tilde f + \frac{n}{v_T^d}\underbrace{\partial_t \tilde f}_{=\frac{\partial \tilde f}{\partial \underline{c}}\frac{\partial c}{\partial t}+\partial_t\tilde f = \frac{\partial\tilde f}{\partial\underline{c}}\cdot\underline{v}\frac{\partial}{\partial t}v_T^{-1}+\partial_t\tilde f}$$

$$= -d\,\frac{n}{v_T^{d+1}}\frac{dv_T}{dt}\tilde f + \frac{n}{v_T^d}\frac{\partial\tilde f}{\partial\underline{c}}\cdot\underline{v}\,(-1)\frac{1}{v_T^2}\frac{dv_T}{dt} + \frac{n}{v_T^d}\partial_t\tilde f(\underline{c},t)$$

$$= -d\,\frac{n}{v_T^{d+1}}\frac{dv_T}{dt}\tilde f - \frac{n}{v_T^{d+1}}\frac{\partial\tilde f}{\partial\underline{c}}\cdot\underline{c}\,\frac{dv_T}{dt} + \frac{n}{v_T^d}\partial_t\tilde f(\underline{c}_1,t)$$

$$= \frac{n}{v_T^{d+1}}\frac{dv_T}{dt}\Big(-d - \underbrace{\underline{c}\,\frac{\partial}{\partial\underline{c}}}_{\text{isotropie}}\Big)\tilde f(c_1,t) + \frac{n}{v_T^d}\frac{\partial}{\partial t}\tilde f(\underline{c}_1,t)$$

$$= -\frac{n}{v_T^{d+1}}\frac{dv_T}{dt}\left(d+c_1\frac{\partial}{\partial c_1}\right)\tilde f(c_1,t) + \frac{n}{v_T^d}\frac{\partial}{\partial t}\tilde f(c_1,t) \qquad (7)$$

$$g_2(\sigma)\, I[f,f] = g_2(\sigma)\,\sigma^{d-1}\int_{\mathbb{R}^d} dv_2\int d\hat\sigma\;\Theta(\hat\sigma\cdot v_{12})(\hat\sigma\cdot v_{12})\left[\frac{1}{\alpha^2}\frac{n^2}{v_T^{2d}}\tilde f(c_1^{**},t)\,\tilde f(c_2^{**},t) - \frac{n^2}{v_T^{2d}}\tilde f(c_1,t)\,\tilde f(c_2,t)\right]$$

$$= g_2(\sigma)\,\sigma^{d-1}\frac{n^2}{v_T^{2d}}\,v_T^d\int_{\mathbb{R}^d} dc_2\int d\hat\sigma\;\Theta(\hat\sigma\cdot c_{12})\,v_T\,(\hat\sigma\cdot c_{12})\left[\frac{1}{\alpha^2}\tilde f(c_1^{**},t)\,\tilde f(c_2^{**},t) - \tilde f(c_1,t)\,\tilde f(c_2,t)\right]$$

; $c = v/v_T$, $dv = v_T\,dc$

$$= g_2(\sigma)\,\sigma^{d-1}\frac{n^2}{v_T^{d-1}}\,\tilde I[\tilde f,\tilde f] \;;\quad \tilde I[\tilde f,\tilde f] = \int_{\mathbb{R}^d} dc_2\int d\hat\sigma\;\Theta(\hat\sigma\cdot c_{12})(\hat\sigma\cdot c_{12})\left[\frac{1}{\alpha^2}\tilde f(c_1^{**},t)\,\tilde f(c_2^{**},t) - \tilde f(c_1,t)\,\tilde f(c_2,t)\right] \qquad (8)$$

$$\frac{\xi_0^2}{2}\frac{\partial^2}{\partial v_1^2} f(v_1,t) = \frac{\xi_0^2}{2}\frac{1}{v_T^2}\frac{\partial^2}{\partial c_1^2}\frac{n}{v_T^d}\tilde f(c_1,t)$$

$$= \frac{\xi_0^2\, n}{2 v_T^{d+2}}\frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t) \qquad (9)$$

(7) à (9) dans (1) donne:

$$-\frac{\cancel{n}}{v_T^{d+1}}\frac{dv_T}{dt}\left(d+c_1\frac{\partial}{\partial c_1}\right)\tilde f(c_1,t) + \frac{\cancel{n}}{v_T^d}\frac{\partial}{\partial t}\tilde f(c_1,t) = g_2(\sigma)\,\sigma^{d-1}\frac{n^{\cancel{2}}}{v_T^{d-1}}\,\tilde I[\tilde f,\tilde f] + \frac{\cancel{n}\,\xi_0^2}{2v_T^{d+2}}\frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t) \qquad (10)$$

$\times\, v_T^{d-1} \Rightarrow$

$$\boxed{-\frac{1}{V_T^2}\frac{dV_T}{dt}\left(d + c_1\frac{\partial}{\partial c_1}\right)\tilde{f}(c_1,t) + \frac{1}{V_T}\frac{\partial}{\partial t}\tilde{f}(c_1,t) = g_2(\sigma)\,\sigma^{d-1} n\, \tilde{I}[\tilde{f},\tilde{f}] + \frac{\xi_0^2}{2V_T^3}\frac{\partial^2}{\partial c_1^2}\tilde{f}(c_1,t)} \quad (11)$$

En divisant tout par $g_2(\sigma) n \sigma^{d-1}$, on retrouve bien l'Éq. (5.25) de la thèse de Noije (concernant le préfacteur du terme en $\partial^2/\partial c_1^2$). On a aussi bien le résultat de Pöschel (il a une erreur de signe). On veut encore éliminer la dérivée dV_T/dt. Pour ceci, on utilise la définition de T (équipartition)

$$\frac{d}{2} n k_B T(t) = \int_{\mathbb{R}^d} dv \, \frac{m}{2} V^2 f(v,t) \quad \Longrightarrow \quad T(t) = \frac{m}{n k_B d}\int_{\mathbb{R}^d} dv\, V^2 f(v,t) \quad (12)$$

Dans cette intégrale, $V = v - u$, où $u = 1/n \int dv\, v\, f(r,v,t)$. Mais comme le système est homogène, f sera tjs pair en v et donc $u = 0$. On peut donc poser $V = v$. On utilise aussi

$$V_T(t) = \sqrt{2/\beta m} = \sqrt{\frac{2 k_B T}{m}} \iff k_B T(t) = \frac{1}{2} m V_T^2(t). \quad (13)$$

On utilise aussi l'expression de la dérivée temporelle d'une fct. $\psi(v_1)$ quelconque : (on aura : $\psi = 1/2\, m v_1^2$)

$$\frac{d}{dt}\langle\psi(t)\rangle = \frac{d}{dt}\int d\underline{v_1}\, \psi(\underline{v_1})\, f(\underline{v_1},t)$$
$$= \int d\underline{v_1}\, \psi(\underline{v_1}) \underbrace{\frac{df}{dt}(v_1,t)}_{= I[f,f] + \frac{\xi_0^2}{2}\frac{\partial^2}{\partial v_1^2} f(v_1,t)}$$
$$= \int dv_1\, \psi(v_1)\, I[f,f] + \frac{\xi_0^2}{2}\int dv_1\, \psi(v_1)\frac{\partial^2}{\partial v_1^2} f(v_1,t)$$
$$= \frac{1}{2} g_2(\sigma)\sigma^{d-1}\int dv_1\int dv_2 \int d\hat{\sigma}\, \Theta(\hat{\sigma}\cdot v_{12})(\hat{\sigma}\cdot v_{12}) f(v_1,t) f(v_2,t)\left[\psi(v_1^*) + \psi(v_2^*) - \psi(v_1) - \psi(v_2)\right]$$
$$+ \underbrace{\frac{\xi_0^2}{2}\int dv_1\, \psi(v_1)\frac{\partial^2}{\partial v_1^2} f(v_1,t)}$$
$$= -\frac{\xi_0^2}{2}\int dv_1 \frac{\partial\psi(v_1)}{\partial v_1}\frac{\partial f(v_1,t)}{\partial v_1} + 0$$
$$= \frac{\xi_0^2}{2}\int dv_1 \frac{\partial^2\psi(v_1)}{\partial v_1^2} f(v_1,t) + 0$$
$$= \frac{1}{2} g_2(\sigma)\sigma^{d-1}\int dv_1\int dv_2\int d\hat{\sigma}\, \Theta(\hat{\sigma}\cdot v_{12})(\hat{\sigma}\cdot v_{12}) f(v_1,t) f(v_2,t)\left[\psi(v_1^*) + \psi(v_2^*) - \psi(v_1) - \psi(v_2)\right]$$
$$+ \frac{\xi_0^2}{2}\int dv_1\, f(v_1,t)\frac{\partial^2\psi(v_1)}{\partial v_1^2} \quad (14)$$

Avec le choix $\psi(v_1) = 1/2\, m v_1^2$, on a

$$\frac{d}{dt}\left\langle \frac{1}{2} m v_1^2\right\rangle = \frac{d}{dt}\frac{1}{2} m \langle v_1^2\rangle$$
$$= \frac{1}{2}\cancel{m}\frac{d}{dt}\frac{n k_B d}{\cancel{m}}\underbrace{\frac{m}{n k_B d}\langle v_1^2\rangle}_{= T(t)}$$
$$= \frac{n k_B d}{2}\frac{dT(t)}{dt} \quad ||$$

$$\stackrel{(14)}{=} \frac{\xi_0^2}{2}\underbrace{\int dv_1\, f(v_1,t)\,\frac{1}{2} m \underbrace{\frac{\partial^2}{\partial v_1^2} v_1^2}_{= \cancel{2}2}}_{= 1/2\, m\cdot 2} \qquad = \frac{\partial}{\partial v_1} 2v_1 = 2$$
$$+ \frac{1}{2} g_2(\sigma)\sigma^{d-1}\int dv_1\int dv_2\int d\hat{\sigma}\,\Theta(\hat{\sigma}\cdot v_{12})(\hat{\sigma}\cdot v_{12}) f(v_1,t) f(v_2,t)\ \frac{1}{2} m \times$$
$$\times\left[\left(v_1 - \frac{1+\alpha}{2}(v_{12}\cdot\hat{\sigma})\hat{\sigma}\right)^2 + \left(v_2 + \frac{1+\alpha}{2}(v_{12}\cdot\hat{\sigma})\hat{\sigma}\right)^2 - v_1^2 - v_2^2\right]$$
$$= 2\frac{m\xi_0^2}{4} + \frac{m}{2}\frac{1}{2} g_2(\sigma)\sigma^{d-1} V_T^{2d}\frac{n^2}{V_T^{2d}} V_T V_T^2 \int dc_1\int dc_2\int d\hat{\sigma}\,\Theta(\hat{\sigma}\cdot c_{12})(\hat{\sigma}\cdot c_{12})\tilde{f}(c_1,t)\tilde{f}(c_2,t)\times$$
$$\times\left[\left(\frac{1+\alpha}{2}\right)^2 (c_{12}\cdot\hat{\sigma})^2\cdot 2 - (1+\alpha)(c_{12}\cdot\hat{\sigma})(c_1\cdot\hat{\sigma}) - (1+\alpha)(v_{12}\cdot\hat{\sigma})(c_2\cdot\hat{\sigma})\right]$$
$$= 2\frac{m\xi_0^2}{4} + \frac{m\, g_2(\sigma)\sigma^{d-1} n^2}{4} V_T^3 \int dc_1 dc_2 \int d\hat{\sigma}\,\Theta(\hat{\sigma}\cdot c_{12})(\hat{\sigma}\cdot c_{12})\tilde{f}(c_1,t)\tilde{f}(c_2,t)\,(1+\alpha)(c_{12}\cdot\hat{\sigma})$$
$$\times\left[(1+\alpha)(c_{12}\cdot\hat{\sigma})/2 - (c_1\cdot\hat{\sigma}) - (c_2\cdot\hat{\sigma})\right]$$

On a donc :

③

$$\frac{dT(t)}{dt} = \frac{m}{nk_Bd}\int dv_1\, v_1^2\, \partial_t f(v_1,t)$$

$$= \frac{m}{nk_Bd}\int dv_1\, v_1^2\, I[f,f] + \frac{m}{nk_Bd}\int dv_1\, v_1^2\, \frac{\xi_0^2}{2}\frac{\partial^2}{\partial v_1^2} f(v_1,t)$$

$$= \frac{m}{nk_Bd}\, g_2(\sigma)\,\sigma^{d-1}\int_{\mathbb{R}^d} dv_1\, v_1^2 \int_{\mathbb{R}^d} dv_2 \int d\hat{\sigma}\, \theta(v_{12}\cdot\hat{\sigma})\,(v_{12}\cdot\hat{\sigma})\left[\frac{1}{\alpha^2} f(v_1^{**},t)\, f(v_2^{**},t) - f(v_1,t)\, f(v_2,t)\right]$$

$$+ \frac{m}{nk_Bd}\,\frac{\xi_0^2}{2}\underbrace{\int dv_1\, v_1^2\, \frac{\partial^2}{\partial v_1^2} f(v_1,t)}$$

$$\overset{\text{partiel}}{=} \int dv_1\, f(v_1,t)\, \frac{\partial^2}{\partial v_1^2} v_1^2$$

$$= \frac{m}{nk_Bd}\, g_2(\sigma)\,\sigma^{d-1}\, \cancel{v_T^{2d}}\, v_T^3\, \frac{n^2}{\cancel{v_T^{2d}}}\int_{\mathbb{R}^d} dc_1\, c_1^2 \int_{\mathbb{R}^d} dc_2 \int d\hat{\sigma}\, \theta(c_{12}\cdot\hat{\sigma})\,(c_{12}\cdot\hat{\sigma})\left[\frac{1}{\alpha^2} f(c_1^{**},t)\, f(c_2^{**},t) - f(c_1,t)\, f(c_2,t)\right]$$

$$+ \frac{m}{nk_Bd}\,\frac{\xi_0^2}{2}\int dv_1\, f(v_1,t) \qquad \underbrace{\frac{\partial}{\partial v_1}\frac{\partial}{\partial v_1}}\sum_{i=1}^{d} v_{1i}^2$$

$$= \underbrace{\nabla_{v_{1j}}\nabla_{v_{1j}}}$$

$$= 2\sum_{i=1}^{d} \underbrace{\nabla_{v_{1j}} v_{1i}}_{=\delta_{ij}}\ \underbrace{\nabla_{v_{1j}} v_{1i}}_{=\delta_{ij}}$$

$$= 2\,\delta_{ij}\,\delta_{ij}$$

$$= 2d$$

$$= \frac{\cancel{m}}{\cancel{nk_B}d}\, g_2(\sigma)\,\sigma^{d-1}\, n^{\cancel{2}}\, v_T\, \frac{2\cancel{k_B}T}{\cancel{m}}\underbrace{\int_{\mathbb{R}^d} dc_1\, c_1^2\, \tilde{I}[\tilde{f},\tilde{f}]}_{=-\mu_2} + \frac{m}{\cancel{nk_B}\cancel{d}}\,\frac{\xi_0^2}{\cancel{2}}\,\cancel{2d}\,\frac{\cancel{n}}{\cancel{v_T^d}}\,\cancel{v_T^d}\underbrace{\int_{\mathbb{R}^d} dc_1\, \tilde{f}(c_1,t)}_{=1}$$

$$= -\frac{2}{d}\, g_2(\sigma)\,\sigma^{d-1} n\, v_T T \mu_2 + \frac{m\,\xi_0^2}{k_B}$$

$$\Longrightarrow \quad \boxed{\frac{dT(t)}{dt} = \frac{m\xi_0^2}{k_B} - \frac{2}{d} B T \mu_2} \quad , \ B = B(t) = v_T(t)\, g_2(\sigma)\,\sigma^{d-1} n \qquad (14)$$

On a bien les mêmes expressions que Pöschel $(\xi_0=0)$ et que Noije (dans le système d'unités $k_B=1$).
L'Eq. (11) devient, avec :

$$\frac{dv_T}{dt} = \frac{d}{dt}\sqrt{\frac{2}{\beta m}} = \frac{d}{dt}\sqrt{\frac{2k_BT}{m}} = \sqrt{\frac{2k_B}{m}}\,\frac{d}{dt}T^{1/2} = \frac{1}{2}\sqrt{\frac{2k_B}{m}}\,\frac{1}{T^{1/2}}\,\frac{dT}{dt}$$

$$= \frac{1}{2T}\, v_T(t)\,\frac{dT(t)}{dt}$$

$$= \frac{v_T}{2T}\left[\frac{m\xi_0^2}{k_B} - \frac{2}{d} B T \mu_2\right]$$

$$= \frac{v_T\, m\, \xi_0^2}{2Tk_B} - \frac{\cancel{2}}{d}\,\frac{v_T}{\cancel{2T}}\, B\cancel{T}\mu_2$$

$$= \frac{1}{2} m\, v_T\, \beta\, \xi_0^2 - \frac{1}{d} v_T B \mu_2 \qquad (15)$$

Ainsi l'Eq. (11) devient :

$$-\frac{1}{V_T}\times\left[\frac{1}{2}m\cancel{V_T}\beta\varepsilon_0^2-\frac{1}{d}\cancel{V_T}BM_2\right]\left(d+c_1\frac{\partial}{\partial c_1}\right)\tilde f(c_1,t)+\frac{1}{V_T}\partial_t\tilde f(c_1,t)=g_2(\sigma)\sigma^{d-1}n\,\tilde I[f,f]+\frac{\varepsilon_0^2}{2V_T^3}\frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t)$$

$V_T=\sqrt{2/\beta m}$; $V_T^2=\frac{\beta m}{2}$

$$\Rightarrow\left[\frac{1}{d}BM_2-\underbrace{\frac{m\beta}{2}}_{=V_T^{-2}}\varepsilon_0^2\right]\left(d+c_1\frac{\partial}{\partial c_1}\right)\tilde f(c_1,t)+\partial_t\tilde f(c_1,t)=g_2(\sigma)\sigma^{d-1}n\,V_T\,\tilde I[\tilde f,\tilde f]+\frac{\varepsilon_0^2}{2V_T^2}\frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t)$$

$$\Rightarrow\left(\frac{M_2}{d}-\frac{1}{BV_T^2}\varepsilon_0^2\right)\left(d+c_1\frac{\partial}{\partial c_1}\right)\tilde f(c_1,t)+B^{-1}\partial_t\tilde f(c_1,t)=\overbrace{\frac{g_2(\sigma)\sigma^{d-1}nV_T}{B}}^{=1}\tilde I[\tilde f,\tilde f]+\frac{\varepsilon_0^2}{2BV_T^2}\frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t)$$

$$\Rightarrow\boxed{\left(\frac{M_2}{d}-\frac{\varepsilon_0^2}{BV_T^2}\right)\left(d+c_1\frac{\partial}{\partial c_1}\right)\tilde f(c_1,t)+\frac{1}{B}\partial_t\tilde f(c_1,t)=\tilde I[\tilde f,\tilde f]+\frac{\varepsilon_0^2}{2BV_T^2}\frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t)}$$

Il s'agit bien de l'Eq. (27) Pöschel, avec $\varepsilon_0\neq 0$.

Suite : méthode de la limite pour le calcul du a_2. Difficulté supplémentaire : $T(t)\neq$ cte.

Par ~~de problème~~ : résoud par $a_2(t)$, puis utilise la solution trouvée pour obtenir $T(t)$.

(1 couplé)

Méthode de la limite : $\lim c_1\to 0$:

$$\left(\frac{M_2}{d}-\frac{\varepsilon_0^2}{BV_T^2}\right)\lim_{c_1\to 0}\left(d+c_1\frac{\partial}{\partial c_1}\right)\tilde f(c_1,t)+\frac{1}{B}\lim_{c_1\to 0}\frac{\partial}{\partial t}\tilde f(c_1,t)=\lim_{c_1\to 0}\tilde I[\tilde f,\tilde f]$$

$$\Rightarrow\left(M_2-\frac{d\,\varepsilon_0^2}{BV_T^2}\right)\tilde f(0,t)+\frac{1}{B}\frac{\partial}{\partial t}\tilde f(0,t)=\underbrace{\lim_{c_1\to 0}\tilde I[\tilde f,\tilde f]}_{\text{déjà calculé !}}+\frac{\varepsilon_0^2}{2BV_T^2}\frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t)\Big|_{c_1=0}\qquad(5)$$

cf. notre article Physica ou thèse (ordre 2 donné)

$$=\tilde I_e+\tilde I_g$$

$$\left\{\begin{array}{ll}\tilde I_e=-\beta_1\tilde f(0,t)\langle c_2\rangle & ;\ \beta_1=\frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}\\[2mm] \quad=-\frac{S_d\tilde M(0)}{2\sqrt{\pi}}\left[1+a_2\frac{d(d+2)}{8}\right]\left(1-\frac{a_2}{8}\right) & ;\ S_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}\quad;\ \tilde M(c)=\pi^{-d/2}\qquad(6)\\[2mm] \tilde I_g=\frac{S_d\tilde M(0)}{2\sqrt{\pi}}\left[\frac{2}{1+\alpha^2}+a_2D_1(\alpha,d)+a_2^2D_2(\alpha,d)\right] & ;\ D_1,D_2:\ \text{p. 133 thèse.}\qquad(7)\\[2mm] \tilde f(c)=\tilde M(c)\left[1+\sum_{i\geq 2}a_i(t)S_i(c^2)\right] & ;\ S_k(x)=\sum_{j=0}^{k}(-1)^j\binom{k+d/2-1}{k-j}\frac{x^j}{j!}\quad;\ M(c)=\pi^{-d/2}e^{-c^2}\qquad(8)\end{array}\right.$$

$$S_2(x)=\frac{1}{2}x^2-\frac{d+2}{2}x+\frac{d(d+2)}{8}$$

Au premier ordre de Sonine :

$$\tilde f(c)=\tilde M(c)\left[1+a_2S_2(c^2)\right]\qquad(9)$$

$$\tilde f(0)=\tilde M(0)\left[1+a_2S_2(0)\right]=\tilde M(0)\left[1+a_2\frac{d(d+2)}{8}\right]\qquad(10)$$

$$\frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t)=\nabla_i\nabla_i\frac{1}{\pi^{d/2}}e^{-c^2}\left[1+a_2\left(\frac{1}{2}c^4-\frac{d+2}{2}c^2+\frac{d(d+2)}{8}\right)\right]\qquad(11)$$

$$=-\tilde f(c_1,t)\,\nabla_i\nabla_i c_jc_j+\tilde M(c)\,a_2\,\nabla_i\nabla_i\left[\frac{1}{2}c_jc_jc_kc_k-\frac{d+2}{2}c_jc_j+\frac{d(d+2)}{8}\right]$$

$$=-\tilde f(c_1,t)\underbrace{2\underbrace{\nabla_i c_j}_{=\delta_{ij}}\underbrace{\nabla_i c_j}_{=\delta_{ij}}}_{=d}+\tilde M(c)\,a_2\left[\frac{1}{2}\nabla_i\nabla_i c_jc_jc_kc_k-\frac{d+2}{2}\underbrace{\nabla_i\nabla_i c_jc_j}_{=2d}\right]$$

$$=-2d\,\tilde f(c_1,t)+\tilde M(c)\,a_2\Big[-d(d+2)+\frac{1}{2}\nabla_i c_jc_kc_k\nabla_i c_j+\frac{1}{2}\nabla_i c_jc_kc_k\nabla_i c_j$$

$$+\frac{1}{2}\nabla_i c_jc_jc_k\nabla_i c_k+\frac{1}{2}\nabla_i c_jc_jc_k\nabla_i c_k\Big]$$

idem

idem

$$= -2d\,\tilde f(c_1,t) + \tilde{\mathcal M}(c_1)\,a_2\left[-d(d+2) + \nabla_i c_j c_k c_k\,\delta_{ij} + \nabla_i c_j c_j c_k\,\delta_{ik}\right]$$

$$= -2d\,\tilde f(c_1,t) + \tilde{\mathcal M}(c_1)\,a_2\Big[-d(d+2) + \delta_{ij} c_k c_k \nabla_i c_j + \delta_{ij} c_j c_k \nabla_i c_k + \delta_{ij} c_j c_k \nabla_i c_k$$
$$+ \delta_{ik} c_j c_k \nabla_i c_j + \delta_{ik} c_j c_k \nabla_i c_j + \delta_{ik} c_j c_j \nabla_i c_k\Big]$$

$$= -2d\,\tilde f(c_1,t) + \tilde{\mathcal M}(c_1)\,a_2\Big[-d(d+2) + c^2 \overbrace{\delta_{ij}\delta_{ij}}^{=d} + \overbrace{\delta_{ij}\delta_{ik} c_j c_k}^{\delta_{ij} c_j \overbrace{\delta_{ik} c_k}^{=c_i} = c^2} + \overbrace{\delta_{ij}\delta_{ik} c_j c_k}^{=\delta_{ij} c_j \overbrace{\delta_{ik} c_k}^{=c_i} = c^2}$$
$$+ \underbrace{\delta_{ik}\delta_{ij} c_j c_k}_{=c^2} + \underbrace{\delta_{ik}\delta_{ij} c_i c_k}_{=c^2} + \underbrace{\delta_{ik}\delta_{ik}}_{=d}\underbrace{c_j c_j}_{=c^2}\Big]$$

$$= -2d\,\tilde f(c_1,t) + \tilde{\mathcal M}(c_1)\,a_2\left[-d(d+2) + c_1^2\,(2d+4)\right]$$

$$= -2d\,\tilde f(c_1,t) + \tilde{\mathcal M}(c_1)\,a_2\left[c_1^2\,2(d+2) - d(d+2)\right]$$

$$= -2d\,\tilde f(c_1,t) + \tilde{\mathcal M}(c_1)\,a_2\,(d+2)\left[2c_1^2 - d\right] \qquad (12)$$

$$\Rightarrow \frac{\partial^2}{\partial c_1^2}\tilde f(c_1,t)\Big|_{c_1\to 0} = -2d\,\tilde f(0,t) + \tilde{\mathcal M}(0)\,a_2(d+2)(-d) \qquad (13)$$

$$\left(\begin{aligned}
&= -2d\,\tilde f(0) - \tilde{\mathcal M}(0)\,a_2\frac{d(d+2)}{8}\cdot 8 + 8\tilde{\mathcal M}(0) - 8\tilde{\mathcal M}(0)\\
&= -2d\,\tilde f(0) \underbrace{- 8\tilde{\mathcal M}(0)\left[1 + a_2\frac{d(d+2)}{8}\right]}_{=\tilde f(0)} + 8\tilde{\mathcal M}(0)\\
&= -(2d+8)\,\tilde f(0) + 8\tilde{\mathcal M}(0)\\
&= -2(d+4)\,\tilde f(0) + 8\tilde{\mathcal M}(0)
\end{aligned}\right) \qquad (14)$$

Conclusion :

$$\left(\mu_2 - \frac{d\xi_0^2}{BV_T^2}\right)\hat f(0,t) + \frac{1}{B}\frac{\partial}{\partial t}\tilde f(0,t) = \tilde I_e + \tilde I_g + \frac{\xi_0^2}{2BV_T^2}\left[-2d\,\tilde f(0,t) - d(d+2)a_2\,\tilde{\mathcal M}(0)\right]$$

$$\Rightarrow \left(\mu_2 - \cancel{\frac{d\xi_0^2}{BV_T^2}} + \cancel{\frac{2d\,\xi_0^2}{2BV_T^2}}\right)\hat f(0,t) + \frac{1}{B}\frac{\partial}{\partial t}\tilde f(0,t) = \tilde I_e + \tilde I_g - \frac{d(d+2)\xi_0^2}{2BV_T^2}\,a_2\,\tilde{\mathcal M}(0)$$

$$\Rightarrow \boxed{\mu_2\,\hat f(0,t) + \frac{1}{B}\frac{\partial}{\partial t}\tilde f(0,t) + \frac{d(d+2)\xi_0^2}{2BV_T^2}\,\tilde{\mathcal M}(0)\,a_2 = \tilde I_e + \tilde I_g} \qquad (15)$$

→ peut être résolu non linéairement.

Admettons-le ici !

$\frac{T}{T_0} = \Theta$; $\Theta = 1 + \delta\Theta$

Explicitement : ~~à l'ordre linéaire en a_2 :~~

$$\mu_2 = \frac{1-\alpha^2}{2}\,\frac{S_d}{\sqrt{2\pi}}\left(1 + \frac{3}{16}a_2\right)$$

$$\tilde f(0,t) = \tilde{\mathcal M}(0)\left[1 + a_2\frac{d(d+2)}{8}\right]$$

NON. cf. p. 9

SUITE p. 9

$$\Rightarrow \frac{1-\alpha^2}{2}\,\frac{S_d}{\sqrt{2\pi}}\left(1 + \frac{3}{16}a_2\right)\tilde{\mathcal M}(0)\left[1 + a_2\frac{d(d+2)}{8}\right] + \frac{1}{B}\dot{\tilde{\mathcal M}}(0)\left[1 + \dot a_2\frac{d(d+2)}{8}\right] + \frac{d(d+2)\xi_0^2}{2BV_T^2}\,\tilde{\mathcal M}(0)\,a_2$$
$$= \tilde I_e + \tilde I_g$$

$\Longrightarrow$ $\qquad a_2\,\frac{2d(d+2)+3}{16}$

$$\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left[1+a_2\frac{d(d+2)}{8}+\frac{3}{16}a_2\right]+\frac{1}{B}\left[1+\dot a_2\frac{d(d+2)}{8}\right]+\frac{d(d+2)\,\xi_0^2}{2BV_T^2}\,a_2=\underbrace{\frac{\tilde I_e}{M(c)}+\frac{\tilde I_g}{M(c)}}+O(a_2^2)$$

$$=\frac{S_d}{2\sqrt{\pi}}\left[\frac{2}{1+\alpha^2}+a_2D_1-1-a_2\frac{d(d+2)}{8}\right]$$

$\Longrightarrow$

$$\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left[1+a_2\frac{2d(d+2)+3}{16}\right]+\frac{1}{B}\left[1+\dot a_2\frac{d(d+2)}{8}\right]+\frac{d(d+2)\,\xi_0^2}{2BV_T^2}\,a_2=\frac{S_d}{2\sqrt{\pi}}\left[\frac{\overbrace{2-1-\alpha^2}^{1-\alpha^2}}{1+\alpha^2}+a_2\left(D_1-\frac{d(d+2)}{8}\right)\right]$$

$\Longrightarrow$

$$1+\dot a_2\frac{d(d+2)}{8}+\frac{d(d+2)\,\xi_0^2}{2V_T^2}\,a_2+B\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}+a_2B\frac{1-\alpha^2}{2}\frac{2d(d+2)+3}{16}-B\frac{S_d}{2\sqrt{\pi}}(1-\alpha^2)$$

$$-\,a_2B\frac{S_d}{2\sqrt{\pi}}\left(D_1-\frac{(d+2)d}{8}\right)=0$$

$$\Longrightarrow\quad \dot a_2(t)\frac{d(d+2)}{8}+a_2(t)\left[\frac{d(d+2)\,\xi_0^2}{2V_T^2}+B\frac{1-\alpha^2}{2}\frac{2d(d+2)+3}{16}-B\frac{S_d}{2\sqrt{\pi}}\left(D_1-\frac{d(d+2)}{8}\right)\right]$$

$$+\,1+B\frac{1-\alpha^2}{\cancel{2}}\frac{S_d}{\sqrt{2\pi}}-B\underbrace{\overbrace{\frac{S_d}{2\sqrt{\pi}}}^{=S_d\left(\frac{\sqrt2}{2}\right)\frac{1}{\sqrt{2\pi}}}}_{=\frac{S_d}{2\sqrt{2\pi}}}(1-\cancel{\alpha^2})=0$$

$$\Longrightarrow\quad \boxed{\dot a_2(t)\frac{d(d+2)}{8}+a_2(t)\left[\frac{d(d+2)\,\xi_0^2}{2V_T^2}+B\frac{1-\alpha^2}{2}\frac{2d(d+2)+3}{16}-B\frac{S_d}{2\sqrt{\pi}}\left(D_1-\frac{d(d+2)}{8}\right)\right]+1=0}\qquad(16)$$

Cette équation est couplée à celle de la température. Ce qu'on veut, $\boxed{T(t)=T_0+\delta T(t)}$
$\Longrightarrow$ on peut linéariser ! De l'Eq. (14) :

$$\frac{dT(t)}{dt}=\frac{m\xi_0^2}{k_B}-\frac{2}{d}BT\mu_2\,,\qquad B=V_T\,g_2(\sigma)\,\sigma^{d-1}\,n$$

$$\mu_2=\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_2(t)\right)$$

$$V_T=\sqrt{\frac{2k_BT}{m}}=\sqrt{\frac{2k_B}{m}}\,T_0\underbrace{\left(1+\frac{\delta T}{T_0}\right)^{1/2}}_{\simeq\,1+\frac12\frac{\delta T}{T_0}}$$

$$=V_T^0\left(1+\frac12\frac{\delta T}{T_0}\right)$$

$$\xi_0^2=(1-\alpha^2)\frac{S_d\,\chi\, n\,\sigma^{d-1}}{m^{3/2}\,d\sqrt{\pi}}(k_BT)^{3/2}$$

$$=\xi_{0,0}^2+\xi_{0,0}^2\,\frac32\,\frac{\delta T(t)}{T_0}$$

$$\Longrightarrow\quad\frac{d\,\delta T(t)}{dt}=\frac{m}{k_B}\xi_{0,0}^2\left(1+\frac32\frac{\delta T(t)}{T_0}\right)-\frac{2}{d}\,g_2(\sigma)\,n\,\sigma^{d-1}\,V_T^0\left(1+\frac12\frac{\delta T}{T_0}\right)T_0\left(1+\frac{\delta T}{T_0}\right)$$

$$\times\,\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_2(t)\right)$$

e même :

$$\boxed{a_2(t)=a_2^s+\delta a_2(t)}$$

$$\Longrightarrow\quad\frac{d\,\delta T(t)}{dt}=\frac{m\,\xi_{0,0}^2}{k_B}+\frac32\frac{m\,\xi_{0,0}^2}{k_B}\frac{\delta T(t)}{T_0}-\frac{2}{d}g_2(\sigma)\,n\,\sigma^{d-1}V_T^0\,T_0\left(1+\frac{\delta T}{T_0}+\frac12\frac{\delta T}{T_0}\right)\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_2(t)\right)$$

$$\Rightarrow \frac{d\delta T(t)}{dt} = \frac{m\xi_{0,0}^2}{k_B} + \frac{3}{2}\frac{m\xi_{0,0}^2}{k_B}\frac{\delta T(t)}{T_0} - \frac{2}{d} g_2(\sigma) n\sigma^{d-1} V_1^0 T_0 \frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{2}\frac{\delta T}{T_0}\right) \times\left(1+\frac{3}{16}a_2^s + \frac{3}{16}\delta a_2(t)\right)$$ (7)

$$= \frac{m\xi_{0,0}^2}{k_B} + \frac{3}{2}\frac{m\xi_{0,0}^2}{k_B}\frac{\delta T(t)}{T_0} - \frac{2}{d} g_2(\sigma) n\sigma^{d-1} V_1^0 T_0 \frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}} \times\left(1+\frac{3}{16}a_2^s + \frac{3}{16}\delta a_2(t) + \frac{3}{2}\frac{\delta T(t)}{T_0}\left(1+\frac{3}{16}a_2^s\right)\right)$$

→ terme stationnaire
→ s'annule avec

$$\boxed{\frac{d\delta T(t)}{dt} = 0 + \frac{3}{2}\frac{m\xi_{0,0}^2}{k_B}\frac{\delta T(t)}{T_0} - \frac{2}{d} B_0 T_0 \frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left[\frac{3}{16}\delta a_2(t) + \frac{3}{2}\left(1+\frac{3}{16}a_2^s\right)\frac{\delta T(t)}{T_0}\right]}$$

De même, par l'équation de a_2 :

$$\delta \dot a_2(t)\frac{d(d+2)}{8} + \left(a_2^s + \delta a_2(t)\right)\left[\frac{d(d+2)}{2}\frac{\xi_0^2}{V_1^2} + B_0\left(1+\frac{1}{2}\frac{\delta T(t)}{T_0}\right)\frac{1-\alpha^2}{2}\frac{2d(d+2)+3}{16} - B_0\left(1+\frac{1}{2}\frac{\delta T(t)}{T_0}\right)\frac{S_d}{2\sqrt{\pi}}\left(D_1 - \frac{d(d+2)}{8}\right)\right] + 1 = 0$$

$$\frac{\xi_0^2}{V_1^2} = (1-\alpha^2)\frac{S_d\,\chi n\sigma^{d+1}}{m^{3/2} d\sqrt{\pi}} k_B^{3/2}\, \underbrace{T^{3/2}}_{=T^{1/2} = T_0^{1/2}\left(1+1/2\,\frac{\delta T(t)}{T_0}\right)} \frac{m}{2k_B T} = \frac{\xi_{0,0}^2}{V_{10}^2}\left(1+\frac{1}{2}\frac{\delta T(t)}{T_0}\right)$$

$\Rightarrow$

$$\delta \dot a_2(t)\frac{d(d+2)}{8} + \left(a_2^s + \delta a_2(t)\right)\left[\overbrace{\frac{d(d+2)}{2}\frac{\xi_{0,0}^2}{V_{10}^2}}^{X}\left(1+\frac{1}{2}\frac{\delta T(t)}{T_0}\right) + B_0\left(1+\frac{1}{2}\frac{\delta T(t)}{T_0}\right)\underbrace{\left\{\frac{1-\alpha^2}{2}\frac{2d(d+2)+3}{16} - \frac{S_d}{2\sqrt{\pi}}\left(D_1 - \frac{d(d+2)}{8}\right)\right\}}_{Y}\right] + 1 = 0$$

$$\Rightarrow \delta \dot a_2(t)\frac{d(d+2)}{8} + \underbrace{\left(a_2^s + \delta a_2(t)\right)\left(1+\frac{1}{2}\frac{\delta T(t)}{T_0}\right)}_{= a_2^s + \frac{1}{2}\frac{\delta T(t)}{T_0}a_2^s + \delta a_2(t)}\left[X + B_0 Y\right] + 1 = 0$$

a_2^s → 0 avec +1, car c'est sol. de l'éq. stationnaire

$$\Rightarrow \delta \dot a_2(t)\frac{d(d+2)}{8} + \left[\frac{1}{2}\frac{\delta T(t)}{T_0}a_2^s + \delta a_2(t)\right]\left[\frac{d(d+2)}{2}\frac{\xi_{0,0}^2}{V_{10}^2} + B_0\left\{\frac{1-\alpha^2}{2}\frac{2d(d+2)+3}{16} - \frac{S_d}{2\sqrt{\pi}}\left(D_1 - \frac{d(d+2)}{8}\right)\right\}\right] = 0$$

$$\Rightarrow \frac{\delta \dot a_2(t)}{a_2^s}\frac{d(d+2)}{8} + \left[\frac{1}{2}\frac{\delta T(t)}{T_0} + \frac{\delta a_2(t)}{a_2}\right]\left[\ \cdots\ \right] = 0$$

Soit $\theta(t) = \delta T(t)/T_0$, $a = \dfrac{\delta a_2(t)}{a_2^s}$, alors :

$$\dot a(t)\frac{d(d+2)}{8} + \left(\frac{1}{2}\theta(t) + a(t)\right)\left[\ \cdots\ \right] = 0$$

En conclusion, on doit résoudre :

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$$\boxed{\begin{aligned} &\dot{a}(t)\,\frac{d(d+2)}{8} + \left[\frac{1}{2}\Theta(t) + a(t)\right]\left[\frac{d(d+2)}{2}\frac{\xi_{010}^2}{V_{T0}^2} + B_0\left\{\frac{1-\alpha^2}{2}\,\frac{2d(d+2)+3}{16} - \frac{S_d}{2\sqrt{\pi}}\left(D_1 - \frac{d(d+2)}{8}\right)\right\}\right] = 0 \\ &\dot{\Theta}(t) = \frac{3}{2}\frac{m\,\xi_{010}^2}{K_B T_0}\,\Theta(t) - \frac{2}{d}\,B_0\,\frac{1-\alpha^2}{2}\,\frac{S_d}{\sqrt{2\pi}}\left[\frac{3}{16}a_2^s\,a(t) + \Theta(t)\,\frac{3}{2}\left(1+\frac{3}{16}a_2'\right)\right]\end{aligned}}$$

Ou ⊕ condensé:

$$\boxed{\begin{aligned} &\dot{a}(t) + \left[\frac{1}{2}\Theta(t) + a(t)\right] C_1 = 0 \\ &\dot{\Theta}(t) + C_2\,\Theta(t) + C_3\,a(t) = 0\end{aligned}}$$

Il aurait fallu adimensionnaliser plus tôt...

Adimensionalisation des équations avant leur linéarisation : $\Theta = T(t)/T_0$; $a = a_2/a_{2s}$; $\theta = 1+\delta\Theta$; $a_2 = 1+\delta a$

- Eq. (14) pour la température :

$$\frac{dT(t)}{dt} = \frac{m\xi_0^2}{k_B} - \frac{2}{d} BT\mu_2 \qquad (1)$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{m\xi_0^2}{k_B T_0} - \frac{2}{d} B\Theta\mu_2 \qquad ; B = V_T g_2 \sigma^{d-1} n \qquad (2)$$

$$\mu_2 = \frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_2\right) + O(a_2^2)$$

$$\xi_0^2 = (1-\alpha^2)\frac{S_d g_2 n \sigma^{d-1}}{m^{3/2} d\sqrt{\pi}}(k_B T)^{3/2}$$

$\mu_2 = \frac{1}{4}(1-\alpha^2)\beta_3 \langle C_{12}^3\rangle_0 \left(1+\frac{3}{16}a_2\right)$

Alors :

$$\frac{m\xi_0^2}{k_B T_0} = \frac{m}{k_B T_0}(1-\alpha^2)\frac{S_d g_2 n\sigma^{d-1}}{m^{3/2} d\sqrt{\pi}} k_B^{3/2}\left(\frac{T^{3/2}}{T_0^{3/2}}\right) T_0^{3/2}$$

$$= (1-\alpha^2)\frac{S_d g_2 n\sigma^{d-1}}{d\sqrt{\pi}}\left(\frac{k_B T_0}{m}\right)^{1/2}\frac{\sqrt{2}}{\sqrt{2}}\,\Theta^{3/2} \qquad ; V_{T0} = \sqrt{\frac{2k_B T_0}{m}}$$

$$= (1-\alpha^2)\frac{S_d g_2 n\sigma^{d-1}}{d\sqrt{2\pi}} V_{T0}\,\Theta^{3/2} \qquad (3)$$

$$\frac{2}{d}B\Theta\mu_2 = \frac{2}{d}\sqrt{\frac{2k_B T}{m}}\sqrt{\frac{T_0}{T_0}}\, g_2\sigma^{d-1} n\,\Theta\,\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a\,a_{20}\right)$$

$$= \frac{2}{d}V_{T0}\, g_2 \sigma^{d-1} n\,\Theta^{3/2}\,\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a\,a_{20}\right) \qquad (4)$$

(3) et (4) dans (2) $\Rightarrow$

$$\frac{d\Theta}{dt} = \frac{S_d g_2 n\sigma^{d-1} V_{T0}}{\sqrt{2\pi}}\frac{1-\alpha^2}{d}\Theta^{3/2} - \frac{S_d g_2 n\sigma^{d-1}V_{T0}}{\sqrt{2\pi}}\frac{2}{d}\Theta^{3/2}\frac{1-\alpha^2}{2}\left(1+\frac{3}{16}a\,a_{20}\right)$$

$$\Rightarrow \frac{1}{\frac{S_d g_2 n\sigma^{d-1}V_{T0}}{\sqrt{2\pi}}}\frac{d\Theta}{dt} = \frac{1-\alpha^2}{d}\Theta^{3/2} - \frac{\cancel{2}}{d}\Theta^{3/2}\frac{1-\alpha^2}{\cancel{2}}\left(1+\frac{3}{16}a\,a_{20}\right) \qquad (5)$$

Soit $\tau = Xt$, avec $X = \frac{S_d g_2 n\sigma^{d-1}V_{T0}}{\sqrt{2\pi}}$, et $d\Theta/d\tau = \dot{\Theta}$, alors : (6)

$$\dot{\Theta} = \cancel{\frac{1-\alpha^2}{d}\Theta^{3/2}} - \cancel{\frac{1-\alpha^2}{d}\Theta^{3/2}} - \frac{1-\alpha^2}{d}\frac{3}{16}a_{20}\,\Theta^{3/2}a$$

$$\Rightarrow \boxed{\dot{\Theta}(t) = -\frac{3a_{20}}{16}\frac{1-\alpha^2}{d}\Theta^{3/2}(t)\,a(t)} \qquad (7)$$

au niveau linéaire (à cause de μ_2)

- Eq. (15) pour a_2 :

$$\mu_2\tilde{f}(0) + \frac{1}{B}\frac{\partial}{\partial t}\tilde{f}(0,t) + \frac{d(d+2)\xi_0^2}{2BV_T^2}\tilde{M}(0)\,a_2 = \tilde{I}_e + \tilde{I}_g \qquad (8)$$

Avec :

$$\mu_2\tilde{f}(0) = \frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a\,a_{20}\right)\tilde{M}(0)\left(1+a\,a_{20}\frac{d(d+2)}{8}\right) \qquad (9)$$

$$\frac{1}{B}\partial_t\tilde{f}(0) = \frac{1}{V_T g_2\sigma^{d-1}n}\frac{\partial}{\partial t}\tilde{M}(0)\left(1+a\,a_{20}\frac{d(d+2)}{8}\right) \qquad ; V_T = \sqrt{\frac{2k_B T_0}{m}}\sqrt{\frac{T}{T_0}} = V_{T0}\Theta^{1/2}$$

$$= \frac{1}{V_{T0}g_2\sigma^{d-1}n}\frac{1}{\Theta^{1/2}}\tilde{M}(0)\frac{\partial}{\partial t}\left[1+a\,a_{20}\frac{d(d+2)}{8}\right] \qquad (10)$$

$$\frac{d(d+2)\xi_0^2}{2BV_T^2}\tilde{M}(0)\,a_2 = \frac{d(d+2)}{2}\tilde{M}(0)\,a\,a_{20}\,\frac{1}{V_{T0}g_2\sigma^{d-1}n}\frac{1}{\Theta^{1/2}}(1-\alpha^2)\frac{S_d g_2 n\sigma^{d-1}}{m^{3/2}d\sqrt{\pi}}(k_B T)^{3/2}\frac{1}{V_T^2}$$

$$= \frac{d(d+2)}{2}\tilde{M}(0)\,a\,a_{20}\frac{1}{\cancel{V_{T0}g_2\sigma^{d-1}n}}\frac{1}{\cancel{\Theta^{1/2}}}(1-\alpha^2)\frac{S_d\cancel{g_2 n\sigma^{d-1}}}{d\sqrt{\pi}}\cancel{\left(\frac{2k_B T_0}{m}\right)^{3/2}}\frac{1}{2}\frac{1}{\sqrt{2}}\cancel{\left(\frac{T}{T_0}\right)^{3/2}}\frac{1}{\cancel{V_{T0}^2}}\cancel{\left(\frac{T_0}{T}\right)^{2/2}}$$

$$= \frac{\cancel{d}(d+2)}{2}\tilde{M}(0)\,a\,a_{20}\frac{(1-\alpha^2)S_d}{2\cancel{d}\sqrt{2\pi}}$$

$$= \frac{d+2}{4}\tilde{M}(0)\,a\,a_{20}(1-\alpha^2)\frac{S_d}{\sqrt{2\pi}} \qquad (11)$$

$$\tilde{I} = \frac{S_d\tilde{M}(0)}{2\sqrt{\pi}}\left[\ \ldots\ \right] \qquad (12)$$

Eqs. (9) à (12) dans (8) $\Rightarrow$

$$\frac{1-\alpha^2}{2}\cancel{\frac{S_d}{\sqrt{2\pi}}}\left(1+\frac{3}{16}aa_{20}\right)\cancel{\tilde{M}(0)}\left(1+aa_{20}\frac{d(d+2)}{8}\right)+\frac{1}{\frac{S_d}{\sqrt{2\pi}}V_{T0}g_2\sigma^{d-1}n}\frac{1}{\Theta^{1/2}}\cancel{\tilde{M}(0)}\frac{\partial}{\partial t}\left[1+aa_{20}\frac{d(d+2)}{8}\right]+\frac{d+2}{4}\cancel{\tilde{M}(0)}aa_{20}(1-\alpha^2)\cancel{\frac{S_d}{\sqrt{2\pi}}}$$

$$=\tau$$

$$=\cancel{\frac{S_d\tilde{M}(0)}{2\sqrt{\pi}}}\left[\dots\right]\sqrt{2\pi}$$

$\Longrightarrow$

$$2\cdot\frac{1-\alpha^2}{2}\left(1+\frac{3}{16}aa_{20}\right)\left(1+aa_{20}\frac{d(d+2)}{8}\right)+2\cdot\frac{1}{\Theta^{1/2}}\frac{\partial}{\partial\tau}\left[1+aa_{20}\frac{d(d+2)}{8}\right]+2\frac{1-\alpha^2}{4}(d+2)aa_{20}=\frac{1}{\sqrt{2}}\left[\dots\right]\cdot 2$$

$\sqrt{2}$

$\Longrightarrow$

$$\frac{1}{\Theta^{1/2}}\dot{a}a_{20}\frac{d(d+2)}{4}+(1-\alpha^2)\left(1+\frac{3}{16}aa_{20}\right)\left(1+aa_{20}\frac{d(d+2)}{8}\right)+\frac{1-\alpha^2}{2}(d+2)aa_{20}=\sqrt{2}\left[\dots\right]$$

$\Longrightarrow$

$$\boxed{\dot{a}(t)a_{20}\frac{d(d+2)}{4}+(1-\alpha^2)\Theta^{1/2}(t)\left[\left(1+\frac{3}{16}aa_{20}\right)\left(1+aa_{20}\frac{d(d+2)}{8}\right)+\frac{d+2}{2}a(t)a_{20}\right]=\sqrt{2}\left[\dots\right]}\qquad(13)$$

A faire :

i) tenir compte de l'ordre non linéaire dans μ_2 ET ξ_0 ! $dT/dt=0 \Longrightarrow \xi_0^2=\frac{k_B}{m}\frac{2}{d}\frac{1}{B}T\mu_2$ ⚠ $\Rightarrow dT/dt=0\ \forall t$

ii) linéariser : $\Theta=1+\delta\Theta$; $a=1+\delta a$.

i) Quel est le choix correct pour ξ_0^2 ? Si on choisit $dT(t)/dt=0\ \forall t$, alors on impose automatiquement la solution $a=\Theta=1\ \forall t$: état stationnaire.
On doit donc faire un autre choix de ξ_0 si on veut une approche à l'équilibre. On doit choisir ξ_0 de sorte à ce que dans l'état stationnaire $dT(t)/dt=0$. Qu'est-ce que cela signifie ?

↳ <u>l'injection d'énergie correspond exactement à ce qu'il faut pour avoir un état stationnaire, mais pas exactement pour compenser les fluctuations de température !</u>

Par contre, il faut que $\omega_E\neq$ cte, car ce terme décrit l'effet des collisions.

Avant cela : il faut déterminer ξ_0^2 et μ_2 de façon cohérente : <u>non linéairement en a_2</u> !!!

$$\frac{dT(t)}{dt}=\frac{m\xi_0^2}{k_B}-\frac{2}{d}BT\mu_2=0$$

$$\Longrightarrow\quad\xi_0^2=\frac{2}{d}\frac{k_B}{m}BT\mu_2\qquad,\quad B=V_T g_2\sigma^{d-1}n$$

Cette valeur de ξ_0^2 est celle de l'état stationnaire, i.e. $T=T_0$, et donc :

$$\xi_0^2=\underbrace{\frac{2}{d}\frac{k_B}{m}V_{T0}g_2\sigma^{d-1}n\,T_0}_{=V_{T0}^2}\mu_2$$

$$=\frac{1}{d}V_{T0}^3g_2\sigma^{d-1}n\,\mu_2$$

Calcul de μ_2 à l'ordre non linéaire :

$$\mu_2=-\int_{\mathbb{R}^d}dc_1\,c_1^2\,\tilde{I}[\tilde{f},\tilde{f}]$$

$$=-\int_{\mathbb{R}^d}dc_1\,c_1^2\int_{\mathbb{R}^d}dc_2\int d\hat{\sigma}\,\Theta(\underline{\sigma}\cdot\underline{c_{12}})(\underline{c_{12}}\cdot\hat{\underline{\sigma}})\left[\frac{1}{\alpha^2}\tilde{f}(c_1^{**})\tilde{f}(c_2^{**})-\tilde{f}(c_1)\tilde{f}(c_2)\right]$$

$$\overset{\text{Pöschel}}{=}\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_2+\frac{9}{1024}a_2^2\right)\qquad(14)$$

Conclusion :

$$\xi_0^2=\overbrace{\frac{1}{d}V_{T0}^3g_2\sigma^{d-1}n\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}}^{=(1-\alpha^2)\frac{S_dg_2n\sigma^{d-1}}{m^{3/2}d\sqrt{\pi}}(k_BT)^{3/2}}\left(1+\frac{3}{16}a_{2s}+\frac{9}{1024}a_{2s}^2\right)\qquad(15)$$

Correction des équations pour tenir compte des non-linéarités :

- Eq. (14) pour la température :

$$\frac{m\xi_0^2}{k_BT_0}=(1-\alpha^2)\frac{S_dg_2n\sigma^{d-1}}{d\sqrt{2\pi}}V_{T0}\left(1+\frac{3}{16}a_{2s}+\frac{9}{1024}a_{2s}^2\right)\qquad(16)$$

$$\frac{2}{d}B\Theta\mu_2=\frac{2}{d}V_{T0}g_2\sigma^{d-1}n\Theta^{3/2}\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}aa_{20}+\frac{9}{1024}a^2a_{20}^2\right)\qquad(17)$$

(16) et (17) dans (2) $\Rightarrow$

$$\dot{\theta} = \frac{1-\alpha^2}{d}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) - \frac{1-\alpha^2}{d}\theta^{3/2}\left(1+\frac{3}{16}a\,a_{20}+\frac{9}{1024}a^2a_{20}^2\right) \quad (18)$$

$$= \frac{1-\alpha^2}{d}\left[1-\theta^{3/2}+\frac{3}{16}a_{20}-\frac{3}{16}\theta^{3/2}a\,a_{20}+\frac{9}{1024}a_{20}^2-\frac{9}{1024}\theta^{3/2}a^2a_{20}^2\right]$$

$$= \frac{1-\alpha^2}{d}\left[1-\theta^{3/2}+\frac{3}{16}a_{20}\left(1-\theta^{3/2}a\right)+\frac{9}{1024}a_{20}^2\left(1-\theta^{3/2}a^2\right)\right] \quad (19)$$

On peut garder plutôt la forme (18). On voit que si $\theta = a = 1$, alors $\dot{\theta} = 0$: ok.

- Eq. (15) pour a_2 :

$$\mu_2\hat{f}(0) = \frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_2+\frac{9}{1024}a_2^2\right)\tilde{M}(0)\left(1+a\,a_{20}\frac{d(d+2)}{8}\right) \quad (20)$$

$$\frac{1}{B}\partial_t\hat{f}(0) = \frac{1}{V_{T0}\,g_2\sigma^{d-1}n}\frac{1}{\theta^{1/2}}\tilde{M}(0)\frac{\partial}{\partial t}\left[1+a\,a_{20}\frac{d(d+2)}{8}\right] \quad (21)$$

$$\frac{d(d+2)\,\varepsilon_0^2}{2B\,V_T^2}\tilde{M}(0)\,a_2 = \frac{d(d+2)}{2}\tilde{M}(0)\,a\,a_{20}\frac{1}{V_{T0}\,g_2\sigma^{d-1}n}\frac{1}{\theta^{1/2}}(1-\alpha^2)\frac{S_d\,g_2\,n\sigma^{d-1}}{m^{3/2}d\sqrt{\pi}}\underbrace{(k_BT_0)^{3/2}}_{=V_{T0}^3}\underbrace{\frac{1}{V_T^2}}_{=1/V_T^2}\left(1+\frac{3}{16}a_2+\frac{9}{1024}a_2^2\right)$$

$$= \frac{\cancel{d}(d+2)}{2}\tilde{M}(0)\,a\,a_{20}\frac{1}{\cancel{V_{T0}}\cancel{g_2\sigma^{d-1}n}}\frac{1}{\theta^{1/2}}(1-\alpha^2)\frac{S_d\,\cancel{g_2 n\sigma^{d-1}}}{2\cancel{d}\sqrt{2\pi}}\left(\frac{2k_BT_0}{m}\right)^{3/2}\frac{m}{2k_BT_0}\underbrace{\frac{T_0}{T}}_{=1/\theta}(\ldots)$$

$$= \frac{d+2}{4}\tilde{M}(0)\,a\,a_{20}\frac{1}{\theta^{3/2}}(1-\alpha^2)\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) \quad (22)$$

$$\tilde{I} = \frac{S_d M(0)}{2\sqrt{\pi}}[\ldots] \quad (23)$$

Eqs. (20) $\leftrightarrow$ (23) dans (8) donne :

$$\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_2+\frac{9}{1024}a_2^2\right)\cancel{\tilde{M}(0)}\left(1+a\,a_{20}\frac{d(d+2)}{8}\right) + \frac{1}{\underbrace{\frac{S_d}{\sqrt{2\pi}}V_{T0}\,g_2\sigma^{d-1}n\,\partial t}_{=\partial\tau}}\frac{1}{\theta^{1/2}}\cancel{\tilde{M}(0)}\,\partial\left[1+a\,a_{20}\frac{d(d+2)}{8}\right]$$

$$+\frac{d+2}{4}\cancel{\tilde{M}(0)}\,a\,a_{20}\frac{1}{\theta^{3/2}}(1-\alpha^2)\frac{S_d}{\sqrt{2\pi}}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) = \frac{S_d\cancel{M(0)}}{\sqrt{2\pi}}\frac{1}{2}[\ldots]$$

$\Rightarrow$

$$\frac{1-\alpha^2}{2}\left(1+\frac{3}{16}a\,a_{20}+\frac{9}{1024}a^2a_{20}^2\right)\left(1+a\,a_{20}\frac{d(d+2)}{8}\right)+\frac{1}{\theta^{1/2}}\dot{a}\,a_{20}\frac{d(d+2)}{8}$$

$$+\frac{d+2}{4}\frac{a\,a_{20}}{\theta^{3/2}}(1-\alpha^2)\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) = \frac{1}{2}[\ldots]$$

$\Rightarrow$

$$\dot{a}(t)\,a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\,\theta^{1/2}(t)\left(1+\frac{3}{16}a(t)a_{20}+\frac{9}{1024}a^2(t)a_{20}^2\right)\left(1+a(t)a_{20}\frac{d(d+2)}{8}\right)$$

$$+(1-\alpha^2)\frac{a(t)a_{20}}{\theta^{3/2}}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\frac{d+2}{2} = [\ldots]$$

Conclusion : $\theta = \frac{T(t)}{T_0} \sim 1+\delta\theta(t)$

$a = a_2(t)/a_{20}$

$$\boxed{\begin{aligned}&\dot{\theta}(\tau) = \frac{1-\alpha^2}{d}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)-\frac{1-\alpha^2}{d}\theta^{3/2}(\tau)\left(1+\frac{3}{16}a(\tau)a_{20}+\frac{9}{1024}a^2(\tau)a_{20}^2\right) && (24)\\ &\dot{a}(\tau)a_{20}\frac{d(d+2)}{4}+(1-\alpha^2)\,\theta^{1/2}(\tau)\left(1+\frac{3}{16}a(\tau)a_{20}+\frac{9}{1024}a^2(\tau)a_{20}^2\right)\left(1+a(\tau)a_{20}\frac{d(d+2)}{8}\right)\\ &\quad+(1-\alpha^2)\frac{a(\tau)a_{20}}{\theta^{3/2}(\tau)}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\frac{d+2}{2} = \tfrac{1}{2}[\ldots] && (25)\end{aligned}}$$

mais $\tilde{I}$ sans préfacteur

2 Eqs. non linéaires couplées.

Cas particuliers :

i) $a = a_{20}\ \forall t$ + linéariser en $\theta = 1+\delta\theta$ $\overset{(24)}{\Rightarrow}$ $\delta\dot{\theta} = 0 - \frac{1-\alpha^2}{d}\frac{3}{2}\delta\theta(t)\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)$

sans linéariser : $\dot{\theta} = C(1-\theta^{3/2})$ ~ idem, ≠ exp., ~ peu + rapide

$\Rightarrow$ $\delta\theta(\tau) = \delta\theta(0)\exp(-C\tau)$: idem avant ? (26)

↳ problème : "irréaliste" du pt. de vue cinétique car on force la distribution des vitesses à être la même $\forall t$

ii) Linéarisation "normale" : $a(\tau) = 1+\delta a(\tau)$

$\theta(\tau) = 1+\delta\theta(\tau)$

suite p. 14

$$\delta\dot\theta = \frac{1-\alpha^2}{d}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) - \frac{1-\alpha^2}{d}\left(1+\frac{3}{2}\delta\theta+\dots\right)\left(1+\frac{3}{16}a_{20}(1+\delta a)+\frac{9}{1024}a_{20}^2(1+2\delta a)\right)+O(\delta^2)$$

$$= \frac{1-\alpha^2}{d}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) - \frac{1-\alpha^2}{d}\left(1+\frac{3}{2}\delta\theta\right)\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2+\frac{3}{16}a_{20}\delta a+\frac{9}{1024}a_{20}^2\delta a\cdot 2\right)$$

$$= 0 - \frac{1-\alpha^2}{d}\left(\frac{3}{16}a_{20}\delta a+\frac{9}{512}a_{20}^2\delta a\right) - \frac{1-\alpha^2}{d}\frac{3}{2}\delta\theta\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)$$

$$= -\frac{1-\alpha^2}{d}\frac{3}{16}a_{20}\,\delta a(\tau)\left(1+\frac{3}{32}a_{20}\right) - \frac{1-\alpha^2}{d}\frac{3}{2}\delta\theta(\tau)\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)$$

$$= -C_{11}\,\delta\theta(\tau) - C_{12}\,\delta a(\tau) \quad ; \quad C_{11} = \frac{1-\alpha^2}{d}\frac{3}{2}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) \qquad (27)$$

$$C_{12} = \frac{1-\alpha^2}{d}\frac{3}{16}a_{20}\left(1+\frac{3}{32}a_{20}\right)$$

Et:

$$\delta\dot a(\tau)\,a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\left(1+\frac{1}{2}\delta\theta(\tau)\right)\left[1+\frac{3}{16}a_{20}(1+\delta a)+\frac{9}{1024}a_{20}^2(1+2\delta a)\right]\left[1+a_{20}\frac{d(d+2)}{8}(1+\delta a)\right]$$
$$+(1-\alpha^2)\,a_{20}(1+\delta a)\left(1-\frac{1}{2}\delta\theta\right)\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\frac{d+2}{2} = r[\dots]$$

$\Longrightarrow$

$$\delta\dot a\,a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\left(1+\frac{1}{2}\delta\theta\right)\left[1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2+\frac{3}{16}a_{20}\delta a+\frac{9}{1024}a_{20}^2\,2\delta a\right]\left[1+a_{20}\frac{d(d+2)}{8}+a_{20}\frac{d(d+2)}{8}\delta a\right]$$
$$+(1-\alpha^2)\,a_{20}(1+\delta a)\left(1-\tfrac{1}{2}\delta\theta\right)\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\frac{d+2}{2}$$
$$= r\left[1+a_{20}(1+\delta a)\frac{d(d+2)}{8}\right]\left(1-\frac{1}{8}a_{20}(1+\delta a)\right) + \frac{2}{1+\alpha^2} + a_{20}(1+\delta a)D_1 + a_{20}^2(1+\delta a/2)D_2$$

$\Longrightarrow$

$$\delta\dot a\,a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\left[1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2+\frac{3}{16}a_{20}\boxed{\delta a}+\frac{9}{1024}a_{20}^2\,2\boxed{\delta a}+\frac{1}{2}\boxed{\delta\theta}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\right]$$
$$\times\left[1+a_{20}\frac{d(d+2)}{8}+a_{20}\frac{d(d+2)}{8}\boxed{\delta a}\right]$$

refaire calcul

$$+(1-\alpha^2)\frac{d+2}{2}a_{20}\left(1-\frac{1}{2}\delta\theta+\delta a\right)$$
$$= \left[1+a_{20}\frac{d(d+2)}{8}+a_{20}\boxed{\delta a}\frac{d(d+2)}{8}\right]\left(1-\frac{1}{8}a_{20}-\frac{1}{8}a_{20}\boxed{\delta a}\right)$$
$$+\frac{2}{1+\alpha^2}+a_{20}D_1+a_{20}^2D_2+a_{20}\boxed{\delta a}D_1+\frac{1}{2}a_{20}^2\boxed{\delta a}D_2$$

$\Longrightarrow$

$$\delta\dot a\,a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\left(1+a_{20}\frac{d(d+2)}{8}\right) O(\delta^0)$$
$$+(1-\alpha^2)\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)a_{20}\frac{d(d+2)}{8}\delta a$$
$$+(1-\alpha^2)\left(\frac{3}{16}a_{20}\delta a+\frac{9}{1024}a_{20}^2\,2\delta a+\frac{1}{2}\delta\theta\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\right)\left(1+a_{20}\frac{d(d+2)}{8}\right)$$
$$+\underbrace{(1-\alpha^2)\frac{d+2}{2}a_{20}}_{O(\delta^0)} + (1-\alpha^2)\frac{d+2}{2}a_{20}\left(\delta a-\frac{1}{2}\delta\theta\right)$$
$$= \left(1+a_{20}\frac{d(d+2)}{8}\right)\left(1-\frac{1}{8}a_{20}\right)^{O(\delta^0)} + \left(1+a_{20}\frac{d(d+2)}{8}\right)\left(-\frac{1}{8}\right)a_{20}\delta a + a_{20}\delta a\frac{d(d+2)}{8}\left(1-\frac{a_{20}}{8}\right)$$
$$+\frac{2}{1+\alpha^2}+a_{20}D_1+a_{20}^2D_2\ {}^{O(\delta^0)} + a_{20}\delta a\,D_1+\frac{1}{2}a_{20}^2\delta a\,D_2$$

Les termes $O(\delta^0)$ doivent s'annuler : c'est la solution stationnaire qui ne contribue pas à $\delta\dot a$. Ainsi :

$$\delta\dot a\,a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\frac{d(d+2)}{8}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)a_{20}\boxed{\delta a}$$
$$+(1-\alpha^2)\left(1+a_{20}\frac{d(d+2)}{8}\right)\left[\left(\frac{3}{16}a_{20}+\overset{3/16}{\frac{9}{512}}a_{20}^2\right)\boxed{\delta a}+\frac{1}{2}\delta\theta\left(\frac{3}{16}a_{20}+\overset{3/64}{\frac{9}{1024}}a_{20}^2\right)\right]$$
$$+(1-\alpha^2)\frac{d+2}{2}a_{20}\boxed{\delta a} - (1-\alpha^2)\frac{d+2}{2}a_{20}\frac{1}{2}\delta\theta$$
$$= -\frac{1}{8}\left(1+a_{20}\frac{d(d+2)}{8}\right)a_{20}\boxed{\delta a} + \frac{d(d+2)}{8}\left(1-\frac{a_{20}}{8}\right)a_{20}\boxed{\delta a} + a_{20}\boxed{\delta a}D_1 + \frac{1}{2}a_{20}^2\boxed{\delta a}D_2$$

$\Longrightarrow$

$$\delta\dot{a}\, \cancel{a_{20}} \frac{d(d+2)}{4} = -\delta a \Bigg[(1-\alpha^2)\frac{d(d+2)}{8}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\cancel{a_{20}} + (1-\alpha^2)\left(1+a_{20}\frac{d(d+2)}{8}\right)\frac{3}{16}\left(\cancel{a_{20}}+\frac{3}{16}a_{20}^{\cancel{2}}\right) + (1-\alpha^2)\frac{d+2}{2}\cancel{a_{20}} + \frac{1}{8}\left(1+a_{20}\frac{d(d+2)}{8}\right)\cancel{a_{20}} - \frac{d(d+2)}{8}\left(1-\frac{a_{20}}{8}\right)\cancel{a_{20}} - \cancel{a_{20}}D_1 - \frac{1}{2}a_{20}^{\cancel{2}}D_2 \Bigg]$$

$$- \delta\theta \Bigg[(1-\alpha^2)\left(1+a_{20}\frac{d(d+2)}{8}\right)\frac{3}{32}\left(\cancel{a_{20}}+\frac{3}{64}a_{20}^{\cancel{2}}\right) - (1-\alpha^2)\frac{d+2}{4}\cancel{a_{20}} \Bigg]$$

$\Longrightarrow$

$$\delta\dot{a} = -\delta a \Bigg[(1-\alpha^2)\frac{1}{2}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) + (1-\alpha^2)\frac{3}{\cancel{16}_4}\frac{\cancel{4}}{d(d+2)}\left(1+a_{20}\frac{d(d+2)}{8}\right)\left(1+\frac{3}{16}a_{20}\right) + (1-\alpha^2)\frac{\cancel{d+2}}{\cancel{2}}\frac{\cancel{4}2}{d\cancel{(d+2)}} + \frac{1}{\cancel{8}_2}\frac{\cancel{4}}{d(d+2)}\left(1+a_{20}\frac{d(d+2)}{8}\right) - \frac{\cancel{d(d+2)}}{2\cancel{8}}\frac{\cancel{4}}{\cancel{d(d+2)}}\left(1-\frac{a_{20}}{8}\right) - \frac{4}{d(d+2)}D_1 - \frac{\cancel{4}2}{d(d+2)}\frac{\cancel{1}}{\cancel{2}}a_{20}D_2 \Bigg]$$

$$- \delta\theta \Bigg[(1-\alpha^2)\frac{\cancel{4}}{d(d+2)}\frac{3}{\cancel{32}_8}\left(1+a_{20}\frac{d(d+2)}{8}\right)\left(1+\frac{3}{64}a_{20}\right) - (1-\alpha^2)\frac{\cancel{4}}{d\cancel{(d+2)}}\frac{\cancel{d+2}}{\cancel{4}} \Bigg]$$

$\Longrightarrow$

$$\delta\dot{a} = -\delta a\,(1-\alpha^2)\Bigg[\frac{1}{2}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) + \frac{3}{4d(d+2)}\left(1+a_{20}\frac{d(d+2)}{8}\right)\left(1+\frac{3}{16}a_{20}\right) + \frac{2}{d} + \frac{1}{2d(d+2)}\frac{1}{1-\alpha^2}\left(1+a_{20}\frac{d(d+2)}{8}\right) - \frac{1}{2}\frac{1}{1-\alpha^2}\left(1-\frac{a_{20}}{8}\right) - \frac{4}{d(d+2)}\frac{1}{1-\alpha^2}D_1 - \frac{2}{d(d+2)}\frac{1}{1-\alpha^2}a_{20}D_2 \Bigg]$$

$$- \delta\theta\,(1-\alpha^2)\Bigg[\frac{3}{8(d+2)d}\left(1+a_{20}\frac{d(d+2)}{8}\right)\left(1+\frac{3}{64}a_{20}\right) - \frac{1}{d} \Bigg]$$

$$= -C_{21}\,\delta\theta - C_{22}\,\delta a$$

Conclusion :

$$\begin{pmatrix} \delta\dot{\theta} \\ \delta\dot{a} \end{pmatrix} = -\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} \delta\theta \\ \delta a \end{pmatrix} \Longrightarrow$$ approche exponentielle à l'équilibre (analyse linéaire !)

Ralentissement non-linéaire ?

Linéarisation:

$$\dot{\delta\theta} = \frac{1-\alpha^2}{d} X - \frac{1-\alpha^2}{d}\left(1+3/2\,\delta\theta\right)\left[1+\frac{3}{16}a_{20}+\frac{3}{16}a_{20}\,\delta a+\frac{9}{1024}a_{20}^2+\frac{9}{1024}a_{20}^2\,2\delta a\right]+O(\delta^2)$$

$$= \frac{1-\alpha^2}{d}X - \frac{1-\alpha^2}{d}\left(1+\frac{3}{2}\delta\theta\right)\left[X + \frac{3}{16}a_{20}\,\delta a + \frac{9}{512}a_{20}^2\,\delta a\right]$$

$$= \frac{1-\alpha^2}{d}X - \frac{1-\alpha^2}{d}\left(1+\frac{3}{2}\delta\theta\right)\left[X + \delta a\, a_{20}\frac{3}{16}\left(1+\frac{3}{32}a_{20}\right)\right]$$

$$= \cancel{\frac{1-\alpha^2}{d}X} - \frac{1-\alpha^2}{d}\left[\cancel{X} + \delta a\,\frac{3}{16}a_{20}\left(1+\frac{3}{32}a_{20}\right)+\frac{3}{2}\delta\theta\, X\right] + O(\delta^2)$$

$$= -\frac{1-\alpha^2}{d}\,\frac{3}{2}X\delta\theta - \frac{1-\alpha^2}{d}\,\frac{3}{16}a_{20}\left(1+\frac{3}{32}a_{20}\right)\delta a$$

$$= -\frac{1-\alpha^2}{d}\,\frac{3}{2}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)\delta\theta - \frac{1-\alpha^2}{d}\,\frac{3}{16}a_{20}\left(1+\frac{3}{32}a_{20}\right)\delta a$$

$$= -C_{11}\,\delta\theta - C_{12}\,\delta a,$$

où :

$$C_{11} = \frac{1-\alpha^2}{d}\,\frac{3}{2}\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right) \quad \checkmark$$

$$C_{12} = \frac{1-\alpha^2}{d}\,\frac{3}{16}a_{20}\left(1+\frac{3}{32}a_{20}\right) \quad \checkmark$$

Seconde équation:

$$\underbrace{\dot{\delta a}\, a_{20}}_{=\dot{\delta a}_2(t)}\frac{d(d+2)}{4} + (1-\alpha^2)\left(1+\frac{1}{2}\delta\theta\right)\left(\underbrace{1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2}_{=X}+\frac{3}{16}a_{20}\,\delta a + \frac{9}{512}a_{20}^2\,\delta a\right)$$

$$\times\left(1+a_{20}\frac{d(d+2)}{8}+a_{20}\frac{d(d+2)}{8}\delta a\right)$$

$$+(1-\alpha^2)\,a_{20}(1+\delta a)(1-\delta\theta)\frac{d+2}{2}\underbrace{\left(1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right)}_{=X}$$

$$= r_2\Bigg\{\underbrace{\frac{1-\alpha^2}{1+\alpha^2} + a_{20}\left[D_1-\frac{d(d+2)-1}{8}\right]+a_{20}^2\left[D_2+\frac{d(d+2)}{64}\right]}_{=Y}$$

$$+a_{20}\delta a\left[D_1-\frac{d(d+2)-1}{8}\right]+\frac{1}{2}\delta a\, a_{20}\left[D_2+\frac{d(d+2)}{64}\right]\Bigg\}$$

$\Longrightarrow$

$$\dot{\delta a}\, a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\left(1+\frac{1}{2}\delta\theta\right)\left[X+\delta a\,\frac{3}{16}a_{20}\left(1+\frac{3}{32}a_{20}\right)\right]\left[1+a_{20}\frac{d(d+2)}{8}+a_{20}\frac{d(d+2)}{8}\delta a\right]$$

$$+(1-\alpha^2)\frac{d+2}{2}X a_{20}\left(1-\delta\theta+\delta a\right) = r_2\left[Y+a_{20}\delta a\left(D_1+\frac{1}{2}D_2-\frac{d(d+2)-1}{8}+\frac{d(d+2)}{128}\right)\right]$$

$\Longrightarrow$

$$\dot{\delta a}\, a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\left(1+\frac{1}{2}\delta\theta\right)\left[X\left(1+a_{20}\frac{d(d+2)}{8}\right)+X a_{20}\frac{d(d+2)}{8}\delta a + \delta a\,\frac{3}{16}a_{20}\left(1+\frac{3}{32}a_{20}\right)\times\left(1+a_{20}\frac{d(d+2)}{8}\right)\right]$$

$$+(1-\alpha^2)\frac{d+2}{2}X a_{20}\left(1-\delta\theta+\delta a\right) = r_2\left[Y+\delta a\, a_{20}\left(D_1+\frac{1}{2}D_2+\frac{-d(d+2)-16d(d+2)+16}{128}\right)\right]$$

$\Longrightarrow$

$$\dot{\delta a}\, a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\left(1+\frac{1}{2}\delta\theta\right)\left[X\left(1+a_{20}\frac{d(d+2)}{8}\right)+\delta a\, a_{20}\left\{X\frac{d(d+2)}{8}+\frac{3}{16}\left(1+\frac{3}{32}a_{20}\right)\left(1+a_{20}\frac{d(d+2)}{8}\right)\right\}\right]$$

$$+(1-\alpha^2)\frac{d+2}{2}X a_{20}\left(1-\delta\theta+\delta a\right) = r_2\left[Y+\delta a\, a_{20}\left(D_1+\frac{1}{2}D_2+\frac{-15\,d(d+2)+16}{128}\right)\right]$$

$\Longrightarrow$

$$\delta\dot a\, a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\Big[X\Big(1+a_{20}\frac{d(d+2)}{8}\Big) + \delta a\, a_{20}\Big\{X\frac{d(d+2)}{8}+\frac{3}{16}\Big(1+\frac{3}{32}a_{20}\Big)\Big(1+a_{20}\frac{d(d+2)}{8}\Big)\Big\}$$
$$+ \frac12\, \delta\theta\, X\Big(1+a_{20}\frac{d(d+2)}{8}\Big)\Big]$$
$$+ (1-\alpha^2)\frac{d+2}{2}\, X a_{20}\,(1-\delta\theta+\delta a) = r_2\Big[\Upsilon + \delta a\, a_{20}\Big(D_1+\frac12 D_2 + \frac{-\cancel{14}^{15}\,d(d+2)+16}{128}\Big)\Big]$$

Or par définition de a_{20}, il doit être tel que l'équation est satisfaite dans l'état stationnaire, i.e. lorsque $\delta\dot a = \delta a = \delta\theta = 0$. On va dire que cela est vrai $\forall a_{20}$, i.e. que l'on ait obtenu a_{20} en linéarisant les équations en a_{20}, ou bien en résolvant le système cubique. Cela n'est pas vraiment une approximation même dans le cas où a_{20} est obtenu par la linéarisation, car dire que l'équation s'annule est ce que devrait idéalement vérifier a_{20}. Donc c'est en quelque sorte une amélioration d'un a_{20} linéarisé que de dire que l'équation non stationnaire par a_{20} et non linéaire s'annule.

On élimine donc tous les termes d'ordre δ^0 $\Longrightarrow$

$$\delta\dot a\, a_{20}\frac{d(d+2)}{4} + (1-\alpha^2)\Big[\delta a\, a_{20}\Big\{X\frac{d(d+2)}{8} + \frac{3}{16}\Big(1+\frac{3}{32}a_{20}\Big)\Big(1+a_{20}\frac{d(d+2)}{8}\Big)\Big\}$$
$$+ \frac12\,\boxed{\delta\theta}\, X \Big(1+a_{20}\frac{d(d+2)}{8}\Big)\Big]$$
$$+ (1-\alpha^2)\frac{d+2}{2}\,X a_{20}\,(\delta a - \boxed{\delta\theta}) = r_2\,\delta a\Big(D_1+\frac12 D_2+\frac{-\cancel{14}^{15}\,d(d+2)+16}{128}\Big)a_{20}$$

$\Longrightarrow$

$$\underbrace{\delta\dot a\,\cancel{a_{20}}}_{=\delta\dot a_2(t)}\frac{d(d+2)}{4} = -(1-\alpha^2)\,\underbrace{\cancel{a_{20}}\Big\{X\frac{d(d+2)}{8}+\frac{3}{16}\Big(1+\frac{3}{32}a_{20}\Big)\Big(1+a_{20}\frac{d(d+2)}{8}\Big)\Big\}\delta a}_{=\delta a_2(t)}$$
$$-(1-\alpha^2)\frac{d+2}{2}\,X\,\underbrace{\cancel{a_{20}}\,\delta a}_{=\delta a_2(t)}$$
$$+ r_2\Big(D_1+\frac12 D_2+\frac{-\cancel{14}^{15}\,d(d+2)+16}{128}\Big)\underbrace{\delta a\,\cancel{a_{20}}}_{\to\,\delta a_2(t)}$$
$$-\cancel{a_{20}}\,\frac{1-\alpha^2}{2}\,X\Big(1+a_{20}\frac{d(d+2)}{8}\Big)\delta\theta$$
$$+(1-\alpha^2)\frac{d+2}{2}\,X\,a_{20}\,\delta\theta$$

$= \delta\dot a_2(t)$ → ne peut plus simplifier ?

ne vient plus ici

reste ici

⚠ division singulière : laisser $a_2(t) = a_{20} + \delta a_2(t)$

$\Longrightarrow$

$$\delta\dot a_2 = -(1-\alpha^2)\Big[X\,\frac{\cancel{d(d+2)}}{2\cancel{8}}\,\frac{\cancel{4}}{\cancel{d(d+2)}} + \frac{3}{\cancel{16}_{4}}\,\frac{\cancel{4}}{d(d+2)}\Big(1+\frac{3}{32}a_{20}\Big)\Big(1+a_{20}\frac{d(d+2)}{8}\Big)\Big]\delta a_2$$
$$-(1-\alpha^2)\,\frac{\cancel{d+2}}{\cancel{2}}\,\frac{\cancel{4}\,2}{d\,\cancel{(d+2)}}\,X\,\delta a_2$$
$$+\frac{r_2\,4}{d(d+2)}\Big[D_1+\frac12 D_2+\frac{-\cancel{14}^{15}\,d(d+2)+16}{128}\Big]\delta a_2$$
$$-\frac{1-\alpha^2}{\cancel{2}\cancel{a_{20}}}\,X\Big(1+a_{20}\frac{d(d+2)}{8}\Big)\delta\theta\,\frac{\cancel{4}\,2}{d(d+2)}$$
$$+(1-\alpha^2)\,\frac{\cancel{d+2}}{\cancel{2}}\,\frac{\cancel{4}\,2}{d\,\cancel{(d+2)}}\,X\,\delta\theta\,a_{20} \qquad , X = 1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2$$

$$\Rightarrow \delta \dot a_2 = -\Bigg\{ (1-\alpha^2)\left[\frac{1}{2}X + \frac{3}{4d(d+2)}\left(1+\frac{3}{32}a_{20}\right)\left(1+a_{20}\frac{d(d+4)}{8}\right)\right] + (1-\alpha^2)\frac{2}{d}X$$

$$- \frac{4r_2}{d(d+2)}\left[D_1 + \frac{1}{2}D_2 - \frac{15d(d+2)+16}{128}\right]\Bigg\}\,\delta a_2$$

$$-\left\{ \frac{(1-\alpha^2)}{\cancel{a_{20}}}\frac{2}{d(d+2)}X\left(1+a_{20}\frac{d(d+2)}{8}\right) - (1-\alpha^2)\frac{2}{d}X \right\}\delta\theta$$

$${}^{\cdot}a_{20}$$

$$= -(1-\alpha^2)\frac{2}{d}X\left[\cancel{\frac{1}{a_{20}(d+2)} + a_{20}\frac{d(d+2)}{8}\frac{1}{a_{20}(d+2)}} - 1\right]\delta\theta$$

$$\frac{1}{d+2} + a_{20}\frac{d(d+2)}{8(d+2)} - a_{20} = \frac{1}{d+2} + a_{20}\left(\frac{d}{8} - \frac{8}{8}\right) = \frac{1}{d+2} + a_{20}\frac{d-8}{8}$$

$$-\left\{ \frac{(1-\alpha^2)}{2d}\left[\frac{1}{2}X + \frac{2}{d}X + \frac{3}{\cancel{4}d(d+2)}\left(1+\frac{3}{32}a_{20}\right)\left(1+a_{20}\frac{d(d+2)}{8}\right)\right]\right.$$

$$\left.- \frac{4r_2}{d(d+2)}\left[D_1 + \frac{1}{2}D_2 - \frac{15d(d+2)+16}{128}\right]\right\}\delta a_2$$

$$= -(1-\alpha^2)\frac{\cancel{2}}{d}X\,\cancel{\frac{1}{a_{20}}\frac{1}{d+2}\left[1+\frac{d-8}{8}a_{20}(d+2)\right]}\;\delta\theta \longrightarrow -(1-\alpha^2)\frac{2}{d}X\,\frac{1}{d+2}\,\frac{1}{\cancel{8}\,4}\left[8+a_{20}(d-8)(d+2)\right]$$

$$= \frac{1}{\cancel{8}\,4}\left[8+(d-8)a_{20}(d+2)\right]$$

$$-\left\{ \frac{1-\alpha^2}{2d}\left[(d+4)X + \frac{3}{2(d+2)}\left(1+\frac{3}{32}a_{20}\right)\left(1+a_{20}\frac{d(d+2)}{8}\right)\right] - \frac{4r_2}{d(d+2)}\left[D_1 + \frac{1}{2}D_2 - \frac{15d(d+2)+16}{128}\right]\right\}\delta a_2$$

$$= -\frac{(1-\alpha^2)\,X}{4d(d+2)\cancel{a_{20}}}\left[8+(d-8)(d+2)a_{20}\right]\delta\theta$$

$$-\left\{ \frac{1-\alpha^2}{4d(d+2)}\left[2(d+2)(d+4)X + 3\left(1+\frac{3}{32}a_{20}\right)\left(1+a_{20}\frac{d(d+2)}{8}\right)\right] - \frac{4r_2}{d(d+2)}\left[D_1+\frac{1}{2}D_2 - \frac{15d(d+2)+16}{128}\right]\right\}\delta a_2$$

Conclusion :

$$\boxed{\begin{pmatrix}\delta\dot\theta\\ \delta\dot a\end{pmatrix} = -\begin{pmatrix} C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}\begin{pmatrix}\delta\theta\\ \delta a\end{pmatrix}}$$

$$C_{11} = \frac{1-\alpha^2}{d}\frac{3}{2}X \qquad , \quad X = 1 + \frac{3}{16}a_{20} + \frac{9}{1024}a_{20}^2$$

$$C_{12} = \frac{1-\alpha^2}{d}\frac{3}{16}a_{20}\left(1+\frac{3}{32}a_{20}\right)$$

$$C_{21} = \frac{1-\alpha^2}{4d(d+2)\cancel{a_{20}}}X\left[8+(d-8)(d+2)a_{20}\right]$$

signe ? a priori positif ⊕ ou

$$C_{22} = \frac{1-\alpha^2}{4d(d+2)}\left[2(d+2)(d+4)X + 3\left(1+\frac{3}{32}a_{20}\right)\left(1+a_{20}\frac{d(d+2)}{8}\right)\right] - \frac{4r_2}{d(d+2)}\left[D_1 + \frac{D_2}{2} - \frac{15d(d+2)+16}{128}\right]$$

C_{11} : positif

Signe ? mettre explicitement le ⊖ ou ⊕ devant, selon le signe

De ces équations, on voit que dans la limite $\alpha\to 1$ on a $C_{11} = C_{12} = 0$, $C_{22} = -\frac{4r_2}{d(d+2)}\left[D_1 + D_2\frac{15d(\ldots)}{\ldots}\right] \neq 0$, et C_{21} a priori $\neq 0$ $\left(\lim_{\alpha\to 1}\frac{1-\alpha^2}{a_{20}} \neq 0\right)$. C'est-à-dire :

$$\delta\dot\theta = -C_{11}\,\delta\theta - C_{12}\,\delta a = 0 \Rightarrow \quad T(t) = \text{cte} \;;\; \delta\theta = 0$$

$$\delta\dot a = -C_{21}\underbrace{\delta\theta}_{=0} - C_{22}\,\delta a = -C_{22}\,\delta a$$

~~signe ?~~

: ~~processus de relaxation de la fct. de distribution vers la maxwellienne~~ → ~~relaxation exponentielle~~, idem. modèle Maxwell.

Détermination de a_{20} :

1) $C_1 \cdot C_2 - C_3 = 0$

2) $C_2 - \frac{C_3}{C_1} = 0$

3) $C_1 - \frac{C_3}{C_2} = 0$

4) $1 - \frac{C_3}{C_1 C_2} = 0$

5) $\frac{C_1 \cdot C_2 - 1}{C_3} = 0$

6) $\frac{C_2}{C_3} - \frac{1}{C_1} = 0$

7) $\frac{C_1}{C_3} - \frac{1}{C_2} = 0 \longrightarrow$ le ⊕ proche du DSMC

8) $\frac{1}{C_3} - \frac{1}{C_1 C_2} = 0$

7 :

$$\frac{(1-\alpha^2)\left[1+\frac{3}{16}a_{20}+\frac{9}{1024}a_{20}^2\right]}{r_2\left\{\frac{1-\alpha^2}{1+\alpha^2}+a_{20}\left[D_1-\frac{d(d+2)-1}{8}\right]+a_{20}^2\left[D_2(\alpha,d)+\frac{d(d+2)}{64}\right]\right.} - \frac{1}{1+a_{20}\frac{(d+4)(d+2)}{8}} = 0$$

$\Rightarrow$

$$r_2(1+\alpha^2)-2 = r_2\left[1+\alpha^2-\underbrace{\frac{2}{r_2}}_{=2\frac{r_2}{2}=r_2}\right] = r_2(1+r_2+\alpha^2)$$

$$a_{20} = \frac{16(\alpha^2-1)(1+\alpha^2)\overbrace{(r_2-2+r_2\alpha^2)}}{96+13r_2+5r_2\alpha^2+(-96+35r_2)\alpha^4+11r_2\alpha^6}$$

$$= -\frac{16(1-\alpha^2)(1+\alpha^2)(1+r_2+\alpha^2)\cancel{r_2}}{\underbrace{\frac{96}{r_2}}_{=\frac{96}{2}r_2}+13\cancel{r_2}+5\cancel{r_2}\alpha^2+(35\cancel{r_2}-\underbrace{\frac{96}{r_2}}_{=48r_2})\alpha^4+11\cancel{r_2}\alpha^6}$$

$$= -\frac{16(1-\alpha^2)(1+\alpha^2)(1+r_2+\alpha^2)}{48r_2+13+5\alpha^2+(35-48r_2)\alpha^4+11\alpha^6}$$

$$= -\frac{16(1-\alpha^2)(1+\alpha^2)(1+r_2+\alpha^2)}{48r_2+13+5\alpha^2-(48r_2-35)\alpha^4+11\alpha^6}$$

$$\sigma_D = k_B \frac{\xi_0^2}{2} \int dv_1 \frac{1}{f(v_1,t)} \left[\nabla_{v_1} f(v_1,t) \right]^2$$

$$; V_T C = v_1 \quad ; f(v,t) = \frac{n}{V_T^d} \tilde{f}(c)$$

$$V_T = \sqrt{\frac{2 k_B T}{m}}$$

$$= k_B \frac{\xi_0^2}{2} \cancel{V_T^d} \int dc \frac{\cancel{V_T^d}}{n} \frac{1}{\tilde{f}(c)} \left[\frac{1}{V_T} \nabla_c \frac{n}{\cancel{V_T^d}} \tilde{f}(c) \right]^2$$

$$= k_B \frac{1}{2} \xi_0^2 \frac{n}{V_T^2} \int dc \frac{1}{\tilde{f}(c)} \left[\nabla_c \tilde{f}(c) \right]^2$$

$$= k_B \frac{1}{2} (1-\alpha^2) \frac{S_d \chi n \sigma^{d-1}}{d \sqrt{\pi}} \left(\frac{k_B T_0}{m} \right)^{3/2} \left(1 + \frac{3}{16} a_2 + \frac{9}{1024} a_2^2 \right) \frac{n}{V_T^2} \int dc \frac{1}{\tilde{f}(c)} \left[\nabla_c \tilde{f}(c) \right]^2$$

$$= k_B \frac{1}{2} (1-\alpha^2) \frac{S_d \chi n \sigma^{d-1}}{d \sqrt{2\pi}} \underbrace{\sqrt{2} \frac{k_B T_0}{m} \sqrt{\frac{k_B T_0}{m}}}_{= V_{T_0}} n \left(\sqrt{\frac{m}{2 k_B T}} \right)^2 (\ldots) \int \ldots$$

$$= k_B \frac{1}{2} (1-\alpha^2) \frac{S_d \chi n^2 \sigma^{d-1}}{d \sqrt{2\pi}} V_{T_0} \frac{\cancel{k_B} T_0}{\cancel{m}} \frac{\cancel{m}}{2 \cancel{k_B} T} (\ldots) \int \ldots$$

$$= k_B \chi \sigma^{d-1} \frac{1}{4} V_{T_0} n^2 (1-\alpha^2) \frac{S_d}{d \sqrt{2\pi}} \frac{T_0}{T} (\ldots) \int \ldots$$

Intégrations numériques de Ioana:

ok, vérifié ①

i) $\int dc_1 \int dc_2 \int d\hat\sigma\ \Theta(c_{12}\cdot\hat\sigma)(c_{12}\cdot\hat\sigma) f_0(c_1) f_0(c_2) \left[S_2(c_1^2) + S_2(c_2^2)\right]\left[\tilde c_1^4 + \tilde c_2^4 - c_1^4 - c_2^4\right]$; $\tilde c_{1,2} = c_{1,2} \mp (c_{12}\cdot\hat\sigma)\hat\sigma$

ii) $\int dc_1 \int dc_2 \int d\hat\sigma\ \Theta(c_{12}\cdot\hat\sigma)(c_{12}\cdot\hat\sigma) f_0(c_1) f_0(c_2) \left[-\frac{d+2}{2}(c_{12}\cdot\hat\sigma)^2 - \{S_2(c_1^2) + S_2(c_2^2)\}(c_{12}\cdot\hat\sigma)^2\right]$; $f_0(c) = \frac{1}{\pi^{d/2}} e^{-c^2}$

iii) $\int dc_1 \int dc_2 \int d\hat\sigma\ \Theta(c_{12}\cdot\hat\sigma)(c_{12}\cdot\hat\sigma) f_0(c_1) f_0(c_2) \left[-\frac12 (c_{12}\cdot\hat\sigma)\{\tilde c_1^2(\tilde c_1\cdot\hat\sigma) - \tilde c_2^2(\tilde c_2\cdot\hat\sigma)\}\right]$

Intégrale i)

Par définition à l'ordre a_2 (la contribution de μ_4 à l'ordre linéaire en a_2):

$$\mu_4^{a_2} = -\frac12 \int dc_1 \int dc_2 \int d\hat\sigma\ \Theta(c_{12}\cdot\hat\sigma)(c_{12}\cdot\hat\sigma) f_0(c_1) f_0(c_2)\left[S_2(c_1^2) + S_2(c_2^2)\right] \underbrace{\Delta(c_1^4 + c_2^4)}_{= \tilde c_1^4 + \tilde c_2^4 - c_1^4 - c_2^4}$$

$$\Rightarrow I_{i)} = -2\mu_4^{a_2}$$

$$\overset{\text{Noije}}{=} -2\beta_3 \langle c_{12}^3\rangle_0 T_2(\alpha = 1)$$; $T_2(\alpha) = \frac{3}{128}(1-\alpha^2)(10d + 39 + 10\alpha^2) + \frac14 (1+\alpha)(d-1)$

$$= -\cancel{2}\frac{d-1}{\cancel{2}}\beta_3\langle c_{12}^3\rangle_0$$ $T_2(\alpha=1) = \frac{d-1}{2}$

$$= \boxed{-\frac{2(d-1)S_d}{\sqrt{2\pi}}}$$ $\beta_3 \langle c_{12}^3\rangle_0 = \frac{2 S_d}{\sqrt{2\pi}}$; $S_d = 2\pi^{d/2}/\Gamma(d/2)$

Intégrale ii)

$$I_{ii)} = -\frac{1}{\pi^d}\int dc_1\int dc_2 \overbrace{\int d\hat\sigma\ \Theta(c_{12}\cdot\hat\sigma)(c_{12}\cdot\hat\sigma)^3}^{=\beta_3 c_{12}^3} e^{-c_1^2} e^{-c_2^2}\left[\frac{d+2}{2} + c^4 + \frac{c_{12}^4}{16} + \frac12 c^2 c_{12}^2 + (c\cdot c_{12})^2 - (d+2)c^2 - \frac{d+2}{4}c_{12}^2 + \frac{d(d+2)}{4}\right]$$

$$= -\frac{1}{\pi^d}\int dc\, e^{-2c^2}\int dc_{12}\, e^{-c_{12}^2/2} c_{12}^3\left[c^4 + \frac{c_{12}^4}{16} + \frac12 c^2 c_{12}^2 + \underbrace{(c\cdot c_{12})^2}_{=\frac1d c^2 c_{12}^2 \text{ :isotropie}} - (d+2)c^2 - \frac{d+2}{4}c_{12}^2 + \frac{d(d+2)}{4} + \frac{d+2}{2}\right]\beta_3$$

A la différence du dernier terme $(d+2)/2$, cette intégrale est la même que celle calculée pour vérifier l'ordre 1 en a_2 de μ_2. On avait alors trouvé (expression correcte car résultat final correct) (c.f. Ⓐ,Ⓑ):

$$\cancel{\frac{1-\alpha^2}{4}}\ \cancel{a_2}\ \frac{1}{\pi^d}\beta_3\int dc\, e^{-2c^2}\int dc_{12}\, e^{-c_{12}^2/2} c_{12}^3\left[c^4 + \frac{c_{12}^4}{16} + \frac12 c^2 c_{12}^2 + \frac1d c^2 c_{12}^2 - (d+2)c^2 - \frac{d+2}{4}c_{12}^2 + \frac{d(d+2)}{4}\right]$$

$$= 4\frac{\cancel{1-\alpha^2}}{2}\ \cancel{a_2}\ \frac{S_d}{\sqrt{2\pi}}\frac{3}{16}$$

$$\Rightarrow -\frac{1}{\pi^d}\int dc\, e^{-2c^2}\int dc_{12}\, e^{-c_{12}^2/2} c_{12}^3\left[c^4 + \frac{c_{12}^2}{16} + \frac12 c^2 c_{12}^2 + \frac1d c^2 c_{12}^2 - (d+2)c^2 - \frac{d+2}{4}c_{12}^2 + \frac{d(d+2)}{4}\right]\beta_3 = -\frac38\frac{S_d}{\sqrt{2\pi}}$$

Le dernier terme donne:

$$-\frac{1}{\cancel{\pi^d}}\underbrace{\int dc\, e^{-2c^2}}_{=\frac{\cancel{\pi^{d/2}}}{2^{d/2}}\frac{\Gamma(d/2)}{\Gamma(d/2)}}\ \underbrace{\int dc_{12}\, e^{-c_{12}^2/2} c_{12}^3}_{=\frac{\cancel{\pi^{d/2}}}{(\frac12)^{\frac{d+3}{2}}}\frac{\Gamma(\frac{d+3}{2})}{\Gamma(d/2)}}\ \underbrace{\beta_3}_{=\pi^{(d-1)/2}\frac{\Gamma(\frac{3+1}{2})}{\cancel{\Gamma(\frac{d+3}{2})}}}\frac{d+2}{2} = -\frac{d+2}{2}\, 2^{\cancel{\frac{d+3}{2}} - \cancel{\frac d2}}\frac{\pi^{d/2}}{\sqrt{\pi}}\underbrace{\frac{\Gamma(2)}{\Gamma(d/2)}}_{\Gamma(2)=1} = -\underbrace{\frac{2\pi^{d/2}}{\Gamma(d/2)}}_{=S_d}\frac{d+2}{2}\frac{\sqrt2}{\sqrt\pi}$$

On a donc:

$$I_{ii)} = -\frac38\frac{S_d}{\sqrt{2\pi}} - (d+2)\frac{S_d}{\sqrt{2\pi}} = -\frac{3 + 8(d+2)}{8}\frac{S_d}{\sqrt{2\pi}} = \boxed{-\frac{8d+19}{8}\frac{S_d}{\sqrt{2\pi}}}$$

Intégrale iii)

$$I_{iii)} = -\frac12\int dc_1\int dc_2\int d\hat\sigma\ \Theta(c_{12}\cdot\hat\sigma)(c_{12}\cdot\hat\sigma)^2 f_0(c_1) f_0(c_2)\left[\tilde c_1^2(\tilde c_1\cdot\hat\sigma) - \tilde c_2^2(\tilde c_2\cdot\hat\sigma)\right]$$

On doit développer le numérateur avec:

$c_1 = c + c_{12}/2$

$c_2 = c - c_{12}/2$

$c_1^2 = c^2 + \frac{c_{12}^2}{4} + 2c\cdot c_{12}$; $\tilde c_1^2 = [c_1 - (c_{12}\cdot\hat\sigma)\hat\sigma]^2 = c_1^2 + (c_{12}\cdot\hat\sigma)^2 - 2(c_{12}\cdot\hat\sigma)(c_1\cdot\hat\sigma)$

$c_2^2 = c^2 + \frac{c_{12}^2}{4} - 2c\cdot c_{12}$; $\tilde c_2^2 = c_2^2 + (c_{12}\cdot\hat\sigma)^2 + 2(c_{12}\cdot\hat\sigma)(c_2\cdot\hat\sigma)$

$c_1\cdot\hat\sigma = c\cdot\hat\sigma + \frac12 c_{12}\cdot\hat\sigma$

$c_2\cdot\hat\sigma = c\cdot\hat\sigma - \frac12 c_{12}\cdot\hat\sigma$

Ainsi:

$$\tilde c_1^2(\tilde c_1\cdot\hat\sigma) - \tilde c_2^2(\tilde c_2\cdot\hat\sigma) = \tilde c_1^2(c_1\cdot\hat\sigma) - \tilde c_2^2(c_2\cdot\hat\sigma) - \tilde c_1^2[(c_{12}\cdot\hat\sigma)\hat\sigma\cdot\hat\sigma] - \tilde c_2^2[(c_{12}\cdot\hat\sigma)\hat\sigma\cdot\hat\sigma]$$

$$= [c_1^2 + (c_{12}\cdot\hat\sigma)^2 - 2(c_{12}\cdot\hat\sigma)(c_1\cdot\hat\sigma)](c_1\cdot\hat\sigma) - [c_1^2 + (c_{12}\cdot\hat\sigma)^2 - 2(c_{12}\cdot\hat\sigma)(c_1\cdot\hat\sigma)](c_{12}\cdot\hat\sigma)$$
$$- [c_2^2 + (c_{12}\cdot\hat\sigma)^2 + 2(c_{12}\cdot\hat\sigma)(c_2\cdot\hat\sigma)](c_2\cdot\hat\sigma) - [c_2^2 + (c_{12}\cdot\hat\sigma)^2 + 2(c_{12}\cdot\hat\sigma)(c_2\cdot\hat\sigma)](c_{12}\cdot\hat\sigma)$$

$$= [c_1^2 + (c_{12}\cdot\hat\sigma)^2 - 2(c_{12}\cdot\hat\sigma)(c_1\cdot\hat\sigma)][(c\cdot\hat\sigma) + \frac12(c_{12}\cdot\hat\sigma) - (c_{12}\cdot\hat\sigma)]$$
$$- [c_2^2 + (c_{12}\cdot\hat\sigma)^2 + 2(c_{12}\cdot\hat\sigma)(c_2\cdot\hat\sigma)][(c\cdot\hat\sigma) - \frac12(c_{12}\cdot\hat\sigma) + (c_{12}\cdot\hat\sigma)]$$

$$= [c_1^2 + (c_{12}\cdot\hat\sigma)^2 - 2(c_{12}\cdot\hat\sigma)(c_1\cdot\hat\sigma)][(c\cdot\hat\sigma) - \frac12(c_{12}\cdot\hat\sigma)]$$
$$- [c_2^2 + (c_{12}\cdot\hat\sigma)^2 + 2(c_{12}\cdot\hat\sigma)(c_2\cdot\hat\sigma)][(c\cdot\hat\sigma) + \frac12(c_{12}\cdot\hat\sigma)]$$

$$= \left[c^2 + \frac{c_{12}^2}{4} + 2(c\cdot c_{12}) \overset{+(c_{12}\cdot\hat\sigma)^2}{-2}(c_{12}\cdot\hat\sigma)(c\cdot\hat\sigma) - \cancel{4}(c_{12}\cancel{\cdot}\hat\sigma)^2\frac{1}{\cancel{4}}\right]\left[(c\cdot\hat\sigma)-\frac12(c_{12}\cdot\hat\sigma)\right]$$

$$-\left[c^2 + \frac{c_{12}^2}{4} - 2(c\cdot c_{12}) \overset{+(c_{12}\cdot\hat\sigma)^2}{+2}(c_{12}\cdot\hat\sigma)(c\cdot\hat\sigma) \mp \cancel{4}(c_{12}\cancel{\cdot}\hat\sigma)^2\left(\cancel{\tfrac{1}{4}}\right)\right]\left[(c\cdot\hat\sigma)+\frac12(c_{12}\cdot\hat\sigma)\right]$$

$$= \left[c^2 + \frac{c_{12}^2}{4} + 2(c\cdot c_{12}) - 2(c_{12}\cdot\hat\sigma)(c\cdot\hat\sigma)\right]\left[(c\cdot\hat\sigma)-\frac12(c_{12}\cdot\hat\sigma)\right]$$

$$-\left[c^2 + \frac{c_{12}^2}{4} - 2(c\cdot c_{12}) + 2(c_{12}\cdot\hat\sigma)(c\cdot\hat\sigma)\right]\left[(c\cdot\hat\sigma)+\frac12(c_{12}\cdot\hat\sigma)\right]$$

$$= \left(\cancel{c^2} + \cancel{\frac{c_{12}^2}{4}} + 2(c\cdot c_{12})\right)(c\cdot\hat\sigma) - 2(c_{12}\cdot\hat\sigma)(c\cdot\hat\sigma)^2$$

$$-\frac12\left(c^2+\frac{c_{12}^2}{4}\right)(c_{12}\cdot\hat\sigma) - (c\cdot c_{12})(c_{12}\cdot\hat\sigma) + (c_{12}\cdot\hat\sigma)^2(c\cdot\hat\sigma)$$

$$-\left(\cancel{c^2} + \cancel{\frac{c_{12}^2}{4}} - 2(c\cdot c_{12})\right)(c\cdot\hat\sigma) \quad 2(c_{12}\cdot\hat\sigma)(c\cdot\hat\sigma)^2$$

$$-\frac12\left(c^2+\frac{c_{12}^2}{4}\right)(c_{12}\cdot\hat\sigma) - (c\cdot c_{12})(c_{12}\cdot\hat\sigma) + (c_{12}\cdot\hat\sigma)^2(c\cdot\hat\sigma)$$

$$= \underbrace{4(c\cdot c_{12})(c\cdot\hat\sigma)}_{\substack{\text{impair en } c_{12}\\ \to 0}} - \left(c^2+\frac{c_{12}^2}{4}\right)(c_{12}\cdot\hat\sigma) - \underbrace{2(c\cdot c_{12})(c_{12}\cdot\hat\sigma)}_{\substack{\text{impair en } c\\ \to 0}} + \underbrace{2(c_{12}\cdot\hat\sigma)^2(c\cdot\hat\sigma)}_{\substack{\text{impair en } c\\ \to 0}} - 4(c_{12}\cdot\hat\sigma)(c\cdot\hat\sigma)^2$$

$$= -\left(c^2+\frac{c_{12}^2}{4}\right)(c_{12}\cdot\hat\sigma) - 4(c_{12}\cdot\hat\sigma)(c\cdot\hat\sigma)^2$$

L'intégrale est alors:

$$I_{iii)} = -\frac12\int dc_1\int dc_2\int d\hat\sigma\ \Theta(c_{12}\cdot\hat\sigma)(c_{12}\cdot\hat\sigma)^2 f_0(c_1)f_0(c_2)\,(-1)\,(c_{12}\cdot\hat\sigma)\left(c^2+\frac{c_{12}^2}{4}\right) + \tilde I \quad : \text{cf. p.} 3$$

$$= +\frac12\,\frac{1}{\pi^d}\int dc\, e^{-2c^2}\int dc_{12}\, e^{-c_{12}^2/2}\, c_{12}^3\left(c^2+\frac{c_{12}^2}{4}\right)\underbrace{\int d\hat\sigma\,\Theta(c_{12}\cdot\hat\sigma)(\hat c_{12}\cdot\hat\sigma)^3}_{=\beta_3} + \tilde I$$

$$= \frac{\beta_3}{2\cancel{\pi^d}}\Bigg[\underbrace{\int dc\, e^{-2c^2}c^2}_{=\frac{\cancel{\pi^{d/2}}}{2^{\frac{d+2}{2}}}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}}\underbrace{\int dc_{12}\, e^{-c_{12}^2/2}c_{12}^3}_{=\frac{\cancel{\pi^{d/2}}}{(\frac12)^{\frac{d+3}{2}}}\frac{\Gamma(\frac{d+3}{2})}{\Gamma(d/2)}} + \frac14\underbrace{\int dc\, e^{-2c^2}}_{=\frac{\cancel{\pi^{d/2}}}{2^{d/2}}\frac{\Gamma(\frac d2)}{\Gamma(\frac d2)}}\underbrace{\int dc_{12}\, e^{-c_{12}^2/2}c_{12}^5}_{=\frac{\cancel{\pi^{d/2}}}{(\frac12)^{\frac{d+5}{2}}}\frac{\Gamma(\frac{d+5}{2})}{\Gamma(d/2)}}\Bigg] + \tilde I$$

$$= \frac{\beta_3}{2}\left[2^{\frac{\cancel{d}+3-\cancel{d}-2}{2}}\,\frac d2\,\frac{d+1}{2}\,\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)} + 2^{\frac{\cancel{d}+5-\cancel{d}-4}{2}}\,\frac{d+3}{2}\,\frac{d+1}{2}\,\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)}\right] + \tilde I$$

$$= \frac{\beta_3}{2}\,\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)}\,\frac{d+1}{2}\left[2^{1/2}\frac d2 + \frac{d+3}{2}\,2^{1/2}\right] + \tilde I \quad ; \ \beta_3 = \pi^{(d-1)/2}\Big/\Gamma\left(\frac{d+3}{2}\right) = \frac{2\pi^{(d-1)/2}}{(d+1)\Gamma(\frac{d+1}{2})}$$

$$= \frac{\pi^{d/2}}{\sqrt\pi}\,\frac{1}{\cancel{(d+1)}}\,\frac{1}{\cancel{\Gamma(\frac{d+1}{2})}}\,\frac{\cancel{\Gamma(\frac{d+1}{2})}}{\Gamma(d/2)}\,\frac{\cancel{d+1}}{2}\,\underbrace{\frac{\sqrt2}{2}}_{=1/\sqrt2}\left[d + (d+3)\right] + \tilde I$$

$$= \underbrace{\frac{2\pi^{d/2}}{2\sqrt{2\pi}\,2}\,\frac{1}{\Gamma(d/2)}}_{=S_d}\left(2d+3\right) + \tilde I \qquad ; \ S_d = 2\pi^{d/2}/\Gamma(d/2)$$

$$= \boxed{\frac{2d+3}{4}\,\frac{S_d}{\sqrt{2\pi}}} + \tilde I \qquad ; \ \tilde I = \frac{S_d}{\sqrt{2\pi}}$$

Conclusion:

$$I = a_{20}\, I_{i)} + \varepsilon\left[I_{ii)} + I_{iii)}\right] \qquad = -4d + 14 - 19 = -4d - 5$$

$$= -a_{20}\,\frac{2(d-1)}{\sqrt{2\pi}}S_d + \varepsilon\frac{S_d}{\sqrt{2\pi}}\left[\frac{4d+6-8d-19+8}{8}\right]$$

$$= -a_{20}\,2(d-1)\frac{S_d}{\sqrt{2\pi}} - \varepsilon\,\frac{4d+5}{8}\,\frac{S_d}{\sqrt{2\pi}}$$

$\Rightarrow$

$$\boxed{\tilde\sigma_{\text{ajust}} - (-\tilde\sigma_0) = -\delta\alpha(t)\left(\frac{\tilde\sigma_0}{2}\right) - \delta a_2(t)\frac{S_d}{\sqrt{2\pi}}\left[a_{20}\,2(d-1) + \varepsilon\,\frac{4d+5}{8}\right]} \qquad ; \ \varepsilon = 1-\alpha$$

Intégration Ioana: dernière intégrale:

$$\int dc\ \frac{1}{\pi^{d/2}} e^{-c^2}\left[2c^6 - 2(d+6)c^4 + \frac{(d+2)(d+8)}{2}c^2\right]$$

$$= \frac{1}{\pi^{d/2}}\pi^{d/2}\left[2\frac{\Gamma(\frac{d+6}{2})}{\Gamma(d/2)} - 2(d+6)\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)} + \frac{(d+2)(d+8)}{2}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}\right]$$

$$= 2\,\frac{d+4}{2}\,\frac{d+2}{2}\,\frac{d}{2} - 2(d+6)\frac{d+2}{2}\,\frac{d}{2} + \frac{(d+2)(d+8)}{2}\,\frac{d}{2}$$

$$= \frac{d+2}{2}\,\frac{d}{2}\left[d+4 - 2(d+6) + d+8\right]$$

$$= \frac{d}{2}\,\frac{d+2}{2}\left[\cancel{d}+4 - 2\cancel{d} - 12 + \cancel{d}\,8\right]$$

$$= 0$$

Conclusion:

$$\tilde{\sigma}_D - \tilde{\sigma}_0 = -\,\delta\Theta(t)\,\tilde{\sigma}_0 + \frac{3}{16}\tilde{\sigma}_0\,\delta a_2(t) + 0$$

Calcul de $\tilde{I}$:

$$\tilde{I} = -\frac{1}{2}\int dc_1\int dc_2\int d\hat{\sigma}\ \Theta(c_{12}\cdot\hat{\sigma})(c_{12}\cdot\hat{\sigma})^2 f_0(c_1) f_0(c_2)(-4)(c_{12}\cdot\hat{\sigma})\, c_i c_j\, \sigma_i\sigma_j$$

$$= 2\,\frac{1}{\pi^d}\underbrace{\int dc\ e^{-2c^2} c_i c_j}_{=\frac{\pi^{d/2}}{2^{\frac{d+2}{2}}}\frac{\cancel{d}}{\cancel{2d}}\frac{\cancel{\Gamma(d/2)}}{\cancel{\Gamma(d/2)}}\delta_{ij}}\int dc_{12}\ e^{-c_{12}^2/2} c_{12}^3\underbrace{\int d\hat{\sigma}\ \Theta(c_{12}\cdot\hat{\sigma})(\hat{c}_{12}\cdot\hat{\sigma})^3\sigma_i\sigma_j}_{=\frac{\beta_3}{d+3}\left(3\hat{c}_{12i}\hat{c}_{12j}+\delta_{ij}\right)}$$

$$= \frac{1}{\pi^d}\,\frac{\pi^{d/2}}{2^{\frac{d+2}{2}}}\,\delta_{ij}\int dc_{12}\ e^{-c_{12}^2/2} c_{12}^3\,\frac{\beta_3}{d+3}\left(3\hat{c}_{12i}\hat{c}_{12j} + \delta_{ij}\right)$$

$$= \frac{1}{\cancel{\pi^d}}\,\frac{\cancel{\pi^{d/2}}}{2^{\frac{d+2}{2}}}\,\frac{\beta_3}{d+3}\,\delta_{ij}\left[3\underbrace{\int dc_{12}\ e^{-c_{12}^2/2} c_{12}\, c_{12i}\, c_{12j}}_{=\frac{\cancel{\pi^{d/2}}}{(\frac{1}{2})^{\frac{d+1+2}{2}}}\frac{d+1}{2d}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)}\delta_{ij}} + \underbrace{\int dc_{12}\ e^{-c_{12}^2/2} c_{12}^3\,\delta_{ij}}_{=\frac{\cancel{\pi^{d/2}}}{(1/2)^{\frac{d+3}{2}}}\frac{\Gamma(\frac{d+3}{2})}{\Gamma(d/2)}}\right]$$

$$= \frac{1}{2^{\frac{d+2}{2}}}\,\frac{\beta_3}{d+3}\,\underbrace{\delta_{ij}\delta_{ij}}_{=d}\,2^{\frac{d+3}{2}}\,\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)}\left[3\frac{d+1}{2d} + \frac{d+1}{2}\right] \qquad ;\ \beta_3 = \frac{2\pi^{(d-1)/2}}{(d+1)\,\Gamma(\frac{d+1}{2})}$$

$$= 2^{\frac{\cancel{d}+3-\cancel{d}-2}{2}}\,\frac{d}{d+3}\,\frac{2\pi^{(d-1)/2}}{\cancel{(d+1)}\,\cancel{\Gamma(\frac{d+1}{2})}}\,\frac{\cancel{\Gamma(\frac{d+1}{2})}}{\Gamma(d/2)}\,\frac{\cancel{d+1}}{2}\left[\frac{3}{d}+1\right]$$

$$= \underbrace{\frac{2^{1/2}}{2}}_{=1/\sqrt{2}}\,\underbrace{\frac{2\pi^{d/2}}{\Gamma(d/2)}}_{=S_d}\,\frac{1}{\sqrt{\pi}}\,\frac{1}{\cancel{d+3}}\,(\cancel{3+d}) \qquad ;\ S_d = 2\pi^{d/2}/\Gamma(d/2)$$

$$= \boxed{\frac{1}{\sqrt{2\pi}}\,S_d}$$

(2) (3)

$$\mu_2 = -\frac12 \frac{1}{\pi^d}\int_{\mathbb{R}^{2d}} dc_1\, dc_2 \int d\hat\sigma\, \Theta(c_{12}\cdot\hat\sigma)(\hat c_{12}\cdot\hat\sigma)\, c_{12}\, e^{-2C^2} e^{-c_{12}^2/2}\, a_2 \left[C^4 + \frac{c_{12}^4}{16} + \frac12 C^2 c_{12}^2 + (C\cdot c_{12})^2 - (d+2)\left(C^2+\frac{c_{12}^2}{4}\right) + \frac{d(d+2)}{4}\right] \times (c_{12}\cdot\hat\sigma)^2\, \frac{\alpha^2-1}{2}$$

$$= \frac{1-\alpha^2}{4}\frac{a_2}{\pi^d}\int_{\mathbb{R}^d} dc_{12}\int_{\mathbb{R}^d} dC\, \underbrace{\int d\hat\sigma\,\Theta(c_{12}\cdot\hat\sigma)(\hat c_{12}\cdot\hat\sigma)^3}_{=\beta_3}\, c_{12}^3\, e^{-2C^2} e^{-c_{12}^2/2}\left[C^4+\frac{c_{12}^4}{16}+\frac12 C^2c_{12}^2 + \underbrace{(C\cdot c_{12})^2}_{=\frac1d C^2 c_{12}^2 \text{ (isotropic)}} - (d+2)\left(C^2+\frac{c_{12}^2}{4}\right) + \frac{d(d+2)}{4}\right]$$

$$= \frac{1-\alpha^2}{4}\frac{a_2}{\pi^d}\beta_3\int_{\mathbb{R}^d} dc_{12}\, e^{-c_{12}^2/2}\, c_{12}^3 \int_{\mathbb{R}^d} dC\, e^{-2C^2}\left[C^4+\frac1{16}c_{12}^4 + \frac{d+2}{2d}C^2c_{12}^2 - (d+2)C^2 - \frac{d+2}{4}c_{12}^2 + \frac{d(d+2)}{4}\right]$$

$$= \frac{1-\alpha^2}{4}\frac{a_2}{\pi^d}\beta_3\Bigg[\int_{\mathbb{R}^d} dc_{12}\, e^{-c_{12}^2/2} c_{12}^3 \int_{\mathbb{R}^d} dC\, e^{-2C^2} C^4 + \frac1{16}\int_{\mathbb{R}^d} dc_{12}\, e^{-c_{12}^2/2} c_{12}^7 \int_{\mathbb{R}^d} dC\, e^{-2C^2}$$
$$+ \frac{d+2}{2d}\int_{\mathbb{R}^d} dc_{12}\, e^{-c_{12}^2/2} c_{12}^5\int_{\mathbb{R}^d} dC\, e^{-2C^2}C^2 - (d+2)\int_{\mathbb{R}^d} dc_{12}\, e^{-c_{12}^2/2}c_{12}^3\int_{\mathbb{R}^d} dC\, e^{-2C^2}C^2$$
$$- \frac{d+2}{4}\int_{\mathbb{R}^d} dc_{12}\, e^{-c_{12}^2/2} c_{12}^5 \int_{\mathbb{R}^d} dC\, e^{-2C^2} + \frac{d(d+2)}{4}\int_{\mathbb{R}^d} dc_{12}\, e^{-c_{12}^2/2}c_{12}^3\int_{\mathbb{R}^d} dC\, e^{-2C^2}\Bigg]$$

$$= \frac{1-\alpha^2}{4}a_2\beta_3\Bigg[2^{\frac{d+3}{2}}2^{-\frac{d+4}{2}}\frac{\Gamma(\frac{d+3}{2})}{\Gamma(d/2)}\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)} + \frac1{16}\, 2^{\frac{d+7}{2}}2^{-d/2}\frac{\Gamma(\frac{d+7}{2})}{\Gamma(d/2)}$$
$$+ \frac{d+2}{2d}2^{\frac{d+5}{2}}2^{-\frac{d+2}{2}}\frac{\Gamma(\frac{d+5}{2})}{\Gamma(\frac d2)}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)} - (d+2)\,2^{\frac{d+3}{2}}2^{-\frac{d+2}{2}}\frac{\Gamma(\frac{d+3}{2})}{\Gamma(d/2)}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}$$
$$- \frac{d+2}{4}2^{\frac{d+5}{2}}2^{-d/2}\frac{\Gamma(\frac{d+5}{2})}{\Gamma(d/2)}\cancel{\frac{\Gamma(d/2)}{\Gamma(d/2)}} + \frac{d(d+2)}{4}2^{\frac{d+3}{2}}2^{-d/2}\frac{\Gamma(\frac{d+3}{2})}{\Gamma(d/2)}\cancel{\frac{\Gamma(d/2)}{\Gamma(d/2)}}\Bigg]$$

$$= \frac{1-\alpha^2}{4}a_2\underbrace{\beta_3}_{=\pi^{(d-1)/2}\frac{\Gamma(\frac{3+1}{2})}{\Gamma(\frac{d+3}{2})}}\Bigg[2^{-1/2}\frac{d+1}{2}\frac{d+2}{2}\frac d2\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)} + \frac1{16}\cancel{2}\, 2^{1/2}\frac{d+5}{\cancel2}\frac{d+3}{\cancel2}\frac{d+1}{\cancel2}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)}$$
$$+ \frac{d+2}{\cancel{2d}}\cancel{2}\,2^{1/2}\frac{d+3}{2}\frac{d+1}{2}\frac{\cancel d}{2}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)} - (d+2)\,2^{1/2}\frac{d+1}{2}\frac d2\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)}$$
$$- \frac{d+2}{\cancel4}\cancel2\, 2^{1/2}\frac{d+3}{2}\frac{d+1}{2}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)} + \frac{d(d+2)}{4}\,2\,2^{1/2}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(d/2)}\frac{d+1}{2}\Bigg]$$

$$= \frac{1-\alpha^2}{4}a_2\,\pi^{(d-1)/2}\frac{1}{\cancel{\Gamma(\frac{d+3}{2})}}\frac{\cancel2}{\cancel{d+1}}\frac{1}{\Gamma(d/2)}\cancel{\Gamma(\tfrac{d+1}{2})}\frac{\cancel{d+1}}{\cancel2}\,2^{1/2}\Bigg[\frac12\frac{d+2}{2}\frac d2 + \frac12\frac{d+5}{2}\frac{d+3}{2} + \frac{d+3}{2}\frac{d+2}{2}$$
$$- \frac{d(d+2)}{2} - \frac{(d+2)(d+3)}{2} + \frac{d(d+2)}{2}\Bigg]$$

$$= \frac{1-\alpha^2}{4}a_2\,\pi^{(d-1)/2}\frac{2^{1/2}}{\Gamma(d/2)}\frac18\Big[d(d+2) + (d+3)(d+5) + \underbrace{2(d+2)(d+3) - \cancel{4(d+2)(d+3)}}_{=-2(d+2)(d+3)}\Big]$$

$$= \frac{1-\alpha^2}{2}a_2\frac{1}{\sqrt{2\pi}}\underbrace{\frac{2\pi^{d/2}}{\Gamma(d/2)}}_{=S_d}\frac1{16}\Big[d(d+2)+(d+3)\{d+5-2d-4\}\Big]$$

$$= a_2\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\frac1{16}\Big[\underbrace{d(d+2)+(d+3)(-d+1)}_{=\cancel{d^2}+\cancel{2d}-\cancel{d^2}+\cancel{d}-\cancel{3d}+3}\Big]$$

$$= a_2\frac{1-\alpha^2}{2}\frac{S_d}{\sqrt{2\pi}}\frac{3}{16}$$

: $\boxed{\text{CORRECT}}$

Ordre a_2^2 : on a :

$$
\begin{aligned}
\Omega(C_1^2)\,\Omega(C_2^2) = \Big\{ & \tfrac{1}{2}\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+(C\cdot C_{12})^2+2(C\cdot C_{12})\big(C^2+\tfrac{C_{12}^2}{4}\big)\Big] - \tfrac{d+2}{2}\Big[C^2+\tfrac{C_{12}^2}{4}+C\cdot C_{12}\Big]+\tfrac{d(d+2)}{8}\Big\} \\
\times \Big\{ & \tfrac{1}{2}\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+(C\cdot C_{12})^2-2(C\cdot C_{12})\big(C^2+\tfrac{C_{12}^2}{4}\big)\Big] - \tfrac{d+2}{2}\Big[C^2+\tfrac{C_{12}^2}{4}-C\cdot C_{12}\Big]+\tfrac{d(d+2)}{8}\Big\} \\
= \tfrac{1}{4}\Big[& \Big(C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+(C\cdot C_{12})^2\Big)^2 + 4\underbrace{(C\cdot C_{12})^2}_{1/d\,C^2C_{12}^2}\big(C^2+\tfrac{C_{12}^2}{4}\big)^2\Big] \\
& -\tfrac{d+2}{4}\Big[C^2+\tfrac{C_{12}^2}{4}-C\cdot C_{12}\Big]\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+(C\cdot C_{12})^2+2(C\cdot C_{12})\big(C^2+\tfrac{C_{12}^2}{4}\big)\Big] \\
& +\tfrac{d(d+2)}{8}\,\tfrac{1}{2}\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+\underbrace{(C\cdot C_{12})^2}_{1/d\,C^2C_{12}^2}+\cancel{2(C\cdot C_{12})\big(C^2+\tfrac{C_{12}^2}{4}\big)}\Big] \\
& -\tfrac{d+2}{4}\Big[C^2+\tfrac{C_{12}^2}{4}+C\cdot C_{12}\Big]\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+(C\cdot C_{12})^2-2(C\cdot C_{12})\big(C^2+\tfrac{C_{12}^2}{4}\big)\Big] \\
& +\tfrac{(d+2)^2}{4}\Big[\big(C^2+C_{12}^2/4\big)^2+\underbrace{(C\cdot C_{12})^2}_{\frac{1}{d}C^2C_{12}^2}\Big] - \tfrac{d(d+2)}{8}\,\tfrac{d+2}{2}\Big[C^2+\tfrac{C_{12}^2}{4}\cancel{+C\cdot C_{12}}\Big] \\
& +\tfrac{d(d+2)}{16}\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+\underbrace{(C\cdot C_{12})^2}_{\frac{1}{d}C^2C_{12}^2}\cancel{-2(C\cdot C_{12})\big(C^2+\tfrac{C_{12}^2}{4}\big)}\Big] \\
& -\tfrac{d+2}{2}\,\tfrac{d(d+2)}{8}\Big[C^2+\tfrac{C_{12}^2}{4}\cancel{-C\cdot C_{12}}\Big] + \Big(\tfrac{d(d+2)}{8}\Big)^2 \\
= \tfrac{1}{4}\Big[& \Big(C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2\Big)^2+(C\cdot C_{12})^4+2\tfrac{1}{d}C^2C_{12}^2\Big(C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2\Big)+\tfrac{4}{d}C^2C_{12}^2\big(C^2+\tfrac{C_{12}^2}{4}\big)^2\Big] \\
\ominus\ & -\tfrac{d+2}{4}\Big[C^2+\tfrac{C_{12}^2}{4}\Big]\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+\tfrac{1}{d}C^2C_{12}^2\Big]+\tfrac{d+2}{4}(C\cdot C_{12})^2\,2\big(C^2+\tfrac{C_{12}^2}{4}\big) \\
& -\tfrac{d+2}{4}\Big[C^2+\tfrac{C_{12}^2}{4}\Big]\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2+\tfrac{1}{d}C^2C_{12}^2\Big]+\tfrac{d+2}{4}\underbrace{(C\cdot C_{12})^2}_{=1/d\,C^2C_{12}^2}\,2\big(C^2+\tfrac{C_{12}^2}{4}\big) \\
& +\tfrac{d(d+2)}{8}\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{d+2}{2d}C^2C_{12}^2\Big] \\
& +\tfrac{(d+2)^2}{4}\Big[\big(C^2+\tfrac{C_{12}^2}{4}\big)^2+\tfrac{1}{d}C^2C_{12}^2\Big]-\tfrac{d(d+2)^2}{16}\big(C^2+\tfrac{C_{12}^2}{4}\big)\overset{\ominus}{}-\tfrac{d(d+2)^2}{16}\big(C^2+\tfrac{C_{12}^2}{4}\big)+\tfrac{d^2(d+2)^2}{64} \\
= \tfrac{1}{4}\Big[& \Big(C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2\Big)^2+(C\cdot C_{12})^4+\tfrac{2}{d}C^2C_{12}^2\Big(C^4+\tfrac{C_{12}^4}{16}+\tfrac{1}{2}C^2C_{12}^2\Big)\Big] \\
& +\tfrac{1}{d}C^2C_{12}^2\big(C^2+\tfrac{C_{12}^2}{4}\big)^2 \\
& -\tfrac{d+2}{2}\Big[C^2+\tfrac{C_{12}^2}{4}\Big]\Big[C^4+\tfrac{C_{12}^4}{16}+\tfrac{d+2}{2d}C^2C_{12}^2-\tfrac{2}{d}C^2C_{12}^2+\tfrac{d(d+2)}{4}\Big] \\
& +\Big[C^4+\tfrac{C_{12}^4}{16}+\underbrace{\tfrac{1}{2}C^2C_{12}^2+\tfrac{1}{d}C^2C_{12}^2}_{=\frac{d+2}{2d}}\Big]\underbrace{\tfrac{d+2}{4}\Big(d+2+\tfrac{d}{2}\Big)}_{=\frac{d+2}{8}(d+2d+4)}+\tfrac{d^2(d+2)^2}{64}
\end{aligned}
$$

Il est préférable de développer ce polynôme à l'aide de Mathematica par éviter les erreurs de calcul :

$$
\mu_2 = a_2^2\,\frac{1-\alpha^2}{4}\,\beta_3\;\underbrace{\frac{1}{\pi^d}\int_{\mathbb{R}^d} dC_{12}\,C_{12}^3\,e^{-C_{12}^2/2}\int_{\mathbb{R}^d} dC\,e^{-2C^2}\sum_{k,l} C_{12}^k\,C^l}_{\text{Mathematica}}
\qquad ;\ \beta_3 = \pi^{(d-1)/2}\,\frac{\Gamma\left(\frac{3+1}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)} = \frac{2\pi^{(d-1)/2}}{(d+1)\,\Gamma\left(\frac{d+1}{2}\right)}
$$